J. Fluid Mech. (2005), *vol.* 539, *pp.* 57–91. © 2005 Cambridge University Press
doi:10.1017/S0022112005005665 Printed in the United Kingdom

Three-wave resonant interactions in unstable media

By NATALYA N. ROMANOVA[1] AND SERGEI Yu. ANNENKOV[2]†

[1]A.M. Obukhov Institute of Atmospheric Physics, Russian Academy of Sciences, Pyzhevsky 3, Moscow 119017, Russia
nata@omega.ifaran.ru

[2]P.P. Shirshov Institute of Oceanology, Russian Academy of Sciences, 36 Nakhimovsky prosp., Moscow 117997, Russia
serge@wave.sio.rssi.ru

(Received 8 September 2003 and in revised form 1 April 2005)

The paper considers the evolution of weakly nonlinear disturbances in linearly unstable stratified shear flows. We develop a generic Hamiltonian formulation for two-dimensional flows. The paper is focused on three-wave resonant interactions, which are always present in the stratified shear flows under consideration as the lowest-order nonlinear process. Two different types of shear flows are considered. The first one is the classical piecewise-linear model with constant density and vorticity in each layer. For such flows, linear instability is due to weak interaction of different modes. The second type is the Kelvin–Helmholtz model, consisting of two layers with different densities and velocities. Velocity shear is assumed to be weakly supercritical. We show that apart from the classical triplets consisting of stable waves, both flow types admit only triplets consisting of one weakly unstable and two neutrally stable waves, and we consider them in detail.

Universal evolution equations for three resonantly interacting wave packets are derived for both cases. For the first flow type, the generic equations coincide with the system derived earlier for a particular case of resonant interactions between unstable and neutral baroclinic waves in a quasi-geostrophic two-layer model. The evolution equations for the Kelvin–Helmholtz model are new, and are studied numerically and analytically in detail. In particular, we demonstrate that resonant interaction with neutral waves can stabilize the growth of the linearly unstable wave. This mechanism is essentially different from the well-known nonlinear stabilization mechanism due to cubic nonlinearity.

1. Introduction

The method of Hamiltonian formalism makes possible the consideration of wave–wave interactions from a universal point of view, not depending on the physical nature of the wave process (Zakharov 1974; Kuznetsov & Zakharov 1997). In recent years, this method has yielded numerous important results in various areas of fluid dynamics. This technique is now well-established for waves in stable media, reducing the diversity

† Present address: Department of Mathematics, Keele University, Keele ST5 5BG, UK; s.annenkov@maths.keele.ac.uk

of weakly nonlinear wave processes to a small number of universal models. Provided that nonlinear interactions are present at the first nonlinear order, all the processes are described in terms of the fundamental model of triplet interactions, which has been extensively studied (Yakubovich 1974; Zakharov 1976; Kaup, Reiman & Bers 1979; Craik 1985).

Another fundamental model, representing the different type of three-wave interactions that occurs in shear flows, was first investigated in the framework of layered fluid models by Cairns (1979) and by Craik & Adam (1979). The existence of positive- and negative-energy waves in the presence of shear can lead to explosive resonant interactions when the amplitudes of all three waves grow simultaneously. Grimshaw (1994) showed that the explosive resonant triads occur also in continuously stratified shear flow. These papers served as a motivation for Vanneste (1996) to study of the interaction of two regular modes and a packet of singular modes.

The present paper aims to extend the method of Hamiltonian formalism to the other class of unstable flows, where instability takes place in the framework of linearized theory. Then, the classical Hamiltonian approach to wave–wave interactions is inapplicable. In particular, in the classical approach a set of normal variables is introduced, in terms of which the quadratic part of the Hamiltonian has an exactly diagonal form. In the presence of linear instability, such variables do not exist. Strictly speaking, this means that the concept of normal variables loses meaning and should be generalized. Moreover, it is intuitively evident, due to energy conservation, that a pair of these 'generalized' normal variables must have the form similar to $(a(\boldsymbol{k}), a(-\boldsymbol{k}))$, $\boldsymbol{k}$ being a wavevector, unlike the classical case where the pair $(a(\boldsymbol{k}), a^*(\boldsymbol{k}))$ is appropriate.

The new canonical variables, introduced in the present paper, allow one to describe the weakly nonlinear dynamics in unstable media (exemplified by shear flows) in a universal way. Clearly, the concept of weak nonlinear interactions prescribes the consideration of weak linear instability only. On the other hand, in a generic situation, it is reasonable to assume that there is only one instability region. Since the instability is weak, this region is small; then, it is sufficient to consider, without the loss of generality, the resonant interaction of one weakly unstable wave and two neutrally stable ones. Using the method of Hamiltonian formalism, we show that this generic situation comprises two different canonical models (and only them). The properties of these models are characterized by the behaviour of two eigenvectors when the corresponding eigenfrequencies coalesce, and only two scenarios are possible.

The first model is represented by the classical piecewise-linear flow with constant density and vorticity in each layer. While velocity is continuous, density and vorticity have jumps at the interfaces. For this type of flow, linear instability of one of the three waves within a resonant triplet is due to the weak coupling of different modes. When the parameter of coupling tends to zero, two different modes at the point of resonance have the same eigenfrequency, but two distinct eigenvectors (Romanova 1998; Grimshaw & Christodoulides 2001). Thus, the eigenfrequency in this case has algebraic and geometric multiplicity both equal to 2.

In the context of a particular problem, resonant interactions of this type were first investigated by Loesch (1974) and Pedlosky (1975). They considered resonant interactions between unstable and neutral inviscid baroclinic waves in a quasi-geostrophic two-layer model on the β-plane. Time-dependent governing equations obtained by Loesch (1974) give the periodical solution for the amplitude of linearly unstable wave, so the nonlinear resonant interactions can stabilize the growth of marginally unstable waves. In this paper, we investigate resonant interaction of a packet of finite-amplitude marginally unstable waves and two packets of neutral waves at the interfaces. We obtain, from the generic viewpoint, a system of evolution equations for amplitudes slowly depending on the spatial and temporal coordinates. In the purely time-dependent case the system of equations coincides with the system obtained by Loesch (1974) for a particular case.

The second type of model, exemplified here by the nonlinear Kelvin–Helmholtz (hereinafter KH) model, consisting of two layers with different densities and velocities. Velocity shear is assumed to be weakly supercritical. Benjamin & Bridges (1997) demonstrated that this problem admits a canonical Hamiltonian formulation. However, weak linear instability in the KH model is essentially different from the instability caused by the weak coupling of modes. It can be said that the KH instability is a single mode instability. The growth rate of unstable waves is determined by the small parameter of supercriticality. If this parameter is equal to zero, the double eigenfrequency corresponds to the single eigenvector. From the viewpoint of linear algebra, the difference between the two models is most clearly demonstrated by the comparison of two 2×2 matrices,

$$\begin{pmatrix} \omega & \varepsilon \\ \varepsilon & \omega \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \omega & 1 \\ \varepsilon^2 & \omega \end{pmatrix},$$

where ε tends to zero. Both matrices lead to the same dispersion equation $(\omega - \lambda)^2 - \varepsilon^2 = 0$ and have the same double eigenvalue ω. However, the eigenvectors of the first matrix remain orthogonal at $\varepsilon = 0$, while those of the second one tend to each other. Thus, in the second model, while the algebraic multiplicity of the eigenfrequency is again equal to 2, its geometric multiplicity now equals unity. It is important to note that in the limit of zero supercriticality, the system remains unstable, but the instability becomes algebraic. We derive a system of evolution equations for the resonant triplet including one unstable and two stable waves, which is essentially different from the system of the first type.

The influence of nonlinear processes on the evolution of weakly unstable disturbances is the fundamental problem of hydrodynamic instability. In particular, the aforementioned phenomenon of explosive instability means that nonlinearity can destabilize waves that are neutral according to linear theory. The inverse process, i.e. nonlinear stabilization of waves that are unstable in the framework of linear theory, has also attracted attention. The most common mechanism of nonlinear stabilization (or amplification) is known from the theory of the Landau equation (e.g. Schmid & Henningson 2001). According to this mechanism, cubic nonlinearity gives rise to a quadratic correction to the linear growth rate, which can be positive or negative, the latter case leading to stabilization. The evolution equation for the dynamics of the weakly unstable wave packet in the KH model was obtained by Weissman (1978) in the form of the nonlinear Klein–Gordon equation. He showed that in this case, the sign of the Landau coefficient depends on the density ratio. In this paper, we consider a different stabilization mechanism that is due to quadratic nonlinearity. Provided that the flow is weakly supercritical, resonant interaction of one unstable wave with two neutrally stable ones can stabilize the algebraic growth of the unstable wave. This quadratic stabilization mechanism acts at the same order as the mechanism based on the cubic self-interaction of the unstable wave.

The paper is organised as follows. Section 2 contains the problem formulation and basic equations. Using semi-Lagrangian variables, the equations are rewritten in the Hamiltonian form, and the expansion of the Hamiltonian in powers of nonlinearity is obtained, up to the fourth order.

In § 3, we consider the three-layer model with constant density and vorticity of each layer in the unperturbed flow. Equations governing dynamics at the interfaces can be written in the Hamiltonian form (Goncharov 1986; Goncharov & Pavlov 1993; Romanova & Yakushkin 2001). We consider the resonant interaction of three wave packets assuming that one of them lies in the region of coupling of modes. Instability is weak if the wavelength of the unstable wave packet is small compared to the distance between the interfaces. Thus, the interaction of one long wave and two short waves is considered, when one of the short waves is weakly unstable. In this case the well-known equations of three-wave nonlinear interaction are not applicable, since the change to normal canonical variables leads to the growth of the nonlinear terms if the eigenfrequencies become close to each other. We use different canonical variables, based on the eigenvectors considered separately without taking into account the weak coupling of modes.

In § 4, the resonant wave interaction is considered in the Kelvin–Helmholtz model, with a flow of inviscid incompressible fluid over a heavier fluid. Velocities of the unperturbed flow in each layer are constant and different. An interfacial surface tension σ is taken into account. If the velocity jump slightly exceeds a certain critical value depending on the density difference between two fluids and the value of surface tension at the interface of the two fluids, small perturbations at the interface are weakly unstable. In this case, we introduce canonical variables in a different way. In the region of wavenumbers where two eigenfrequencies are close to each other, the canonical variables are based on the pair of vectors that have the sense of the eigenvector and the generalized eigenvector. In these variables, the canonical form of the equations has another structure, and the evolution equations are different from the equations of the preceding section. In particular, they contain the cubic term characterizing self-interaction of the unstable wave that is of the same order as the quadratic terms responsible for the resonant interaction.

In § 5, we consider analytical and numerical solutions of this system for an important particular case. We demonstrate that the effects of interaction with neutral waves can stabilize the growth of the linearly unstable wave. If the initial amplitudes of two neutral waves are much smaller than the initial amplitude of the linearly unstable wave, this wave is shown to be bounded. Its amplitude evolves as a sequence of increases and decreases of parabolic form, with stochastic maxima. We show that this is due to the existence of the intermediate adiabatic integral of motion, which is conserved with great accuracy when this amplitude is not small, and changes randomly when it approaches zero.

Section 6 contains concluding remarks. A derivation of the expressions for interaction coefficients is performed in the Appendix.

2. Formulation

2.1. *Basic equations*

We consider two-dimensional dynamics of ideal incompressible stratified fluid, laterally and vertically unbounded. In the Boussinesq approximation, the basic equations have the form

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + w\frac{\partial \rho}{\partial z} = 0, \qquad (2.1a)$$

$$\frac{\partial \Omega}{\partial t} + u\frac{\partial \Omega}{\partial x} + w\frac{\partial \Omega}{\partial z} - \frac{g}{\rho}\frac{\partial \rho}{\partial x} = 0, \qquad (2.1b)$$

where $\boldsymbol{V} = \{u, w\}$ is the velocity vector, $\Omega = \mathrm{curl}\boldsymbol{V}$ is the vorticity component orthogonal to the plane (x, z), x and z are the horizontal and vertical coordinates, respectively, ρ is the fluid density and g is the acceleration due to gravity.

The horizontal and vertical velocity components u, w are $u = -\partial_z \Psi$, $w = \partial_x \Psi$, where Ψ is the stream function subject to the Poisson equation $\Delta\Psi = -\Omega$. Wave disturbances are assumed to decay as $z \to \pm\infty$. A solution to this equation can be presented in the form

$$\Psi = -\int \Omega(x', z')\mathscr{G}(x - x', z - z')\,\mathrm{d}x'\,\mathrm{d}z',$$

where

$$\mathscr{G}(x - x', z - z') = \frac{1}{4\pi}\ln[(x - x')^2 + (z - z')^2] \tag{2.2}$$

is the Green's function for the Laplace operator. Integration here and below is performed from $-\infty$ to $+\infty$.

2.2. *Semi-Lagrangian coordinates*

Following Romanova & Yakushkin (2001), we rewrite (2.1a,b) in semi-Lagrangian coordinates, introducing Lagrangian coordinate h,

$$z = s(x, t, h) = h + \eta(x, h, t), \tag{2.3}$$

so that the equation

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + w\frac{\partial h}{\partial z} = 0$$

holds at any surface where density is constant.

System (2.1a,b) in coordinates x, h takes the form

$$\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = w, \tag{2.4a}$$

$$\left(\frac{\partial \Omega}{\partial t} + u\frac{\partial \Omega}{\partial x}\right)\frac{\partial s}{\partial h} - N^2\frac{\partial s}{\partial x} = 0, \tag{2.4b}$$

where the Brunt–Väisälä frequency N is defined as

$$N^2 = -\frac{g}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}h}. \tag{2.5}$$

In semi-Lagrangian coordinates, the vorticity density $\tilde{\Omega}(x, h, t) = \Omega(x, s, t)(1+\partial\eta/\partial h)$ satisfies the equation

$$\frac{\partial \tilde{\Omega}}{\partial t} + \frac{\partial (u\tilde{\Omega})}{\partial x} - N^2\frac{\partial s}{\partial x} = 0, \tag{2.6}$$

and the energy, normalized by the averaged density ρ_0, is

$$\begin{aligned} H &= \int \left(\frac{1}{2}\Psi(x, z, t)\Omega(x, z, t) + gz\frac{\rho}{\rho_0}\right)\mathrm{d}x\,\mathrm{d}z \\ &= -\frac{1}{2}\int \Omega(x, z)\Omega(x', z')\mathscr{G}(x - x', z - z')\,\mathrm{d}x'\,\mathrm{d}z'\,\mathrm{d}x\,\mathrm{d}z + \int gz\frac{\rho}{\rho_0}\,\mathrm{d}x\,\mathrm{d}z \\ &= -\frac{1}{2}\int \tilde{\Omega}(x, h)\tilde{\Omega}(x', h')\mathscr{G}(x - x', s(x, h) - s(x', h'))\,\mathrm{d}x'\,\mathrm{d}h'\,\mathrm{d}x\,\mathrm{d}h \\ &\quad + \int gs\frac{\rho}{\rho_0}\frac{\partial s}{\partial h}\,\mathrm{d}x\,\mathrm{d}h. \end{aligned} \tag{2.7}$$

2.3. *Hamiltonian formulation*

Our next step is to present the basic equations in the Hamiltonian form. Following Romanova & Yakushkin (2001), let us introduce the new dependent variable ϕ as

$$\tilde{\Omega}(x,h,t) = \nu(h) + \frac{\partial(\nu\eta)}{\partial h} - \frac{\partial\phi}{\partial x}, \tag{2.8}$$

where η is defined in (2.3), and $\nu(h)$ is the vorticity of the unperturbed flow. In terms of the new variables η, ϕ, (2.4*a*) and (2.6) can be written in the Hamiltonian form (see Romanova & Yakushkin 2001 for details) as

$$\frac{\partial}{\partial x}\dot{\phi}(x,t,h) = -\frac{\partial}{\partial x}\frac{\delta H}{\delta\eta(x,t,h)} + \nu_h\frac{\delta H}{\delta\phi(x,t,h)}, \quad \dot{\eta}(x,t,h) = \frac{\delta H}{\delta\phi(x,t,h)}, \tag{2.9}$$

where a dot denotes the time derivative, ν_h is the derivative of ν with respect to h, and the Hamiltonian H is defined by (2.7).

Then, the normalized kinetic energy E has the form

$$E = -\frac{1}{2}\int\left[\nu + \frac{\partial(\nu\eta)}{\partial h} - \frac{\partial\phi}{\partial x}\right]\left[\nu' + \frac{\partial(\nu'\eta')}{\partial h'} - \frac{\partial\phi'}{\partial x'}\right]\mathscr{G}(x-x',s-s')\,\mathrm{d}x'\,\mathrm{d}h'\,\mathrm{d}x\,\mathrm{d}h, \tag{2.10}$$

where the Green's function $\mathscr{G}$ defined by (2.2) can be written in the form

$$\mathscr{G}(x-x',s-s') = -\frac{1}{4\pi}\int\frac{1}{|k|}\exp(\mathrm{i}k(x-x') - |k||(h-h') + (\eta-\eta')|)\,\mathrm{d}k, \tag{2.11}$$

and we have used the compact notation $\nu' = \nu(h')$, $\eta' = \eta(x',h')$. In order to obtain the expansion of the Hamiltonian in powers of ϕ and η, we expand $\mathscr{G}$ in powers of η:

$$\mathscr{G} = G + G_h(\eta-\eta') + G_{hh}(\eta-\eta')^2/2 + \ldots,$$

where

$$G = -\frac{1}{4\pi}\int\frac{1}{|k|}\exp[\mathrm{i}k(x-x') - |k||h-h'|]\,\mathrm{d}k,$$

$$G_h = \frac{1}{4\pi}\mathrm{sign}\,(h-h')\int\exp[\mathrm{i}k(x-x') - |k||h-h'|]\,\mathrm{d}k.$$

Higher terms in the expansion can be obtained using the equation

$$G_{hh} + G_{xx} = \delta(h-h')\delta(x-x').$$

Then, the leading-order term of the expansion of the Hamiltonian in powers of ϕ, η is

$$H_2 = \frac{1}{2}\int\{\phi S[\phi] + 2V(h)\eta\phi_x + [N^2 - V(h)\nu_h(h)]\eta^2\}\,\mathrm{d}x\,\mathrm{d}h, \tag{2.12}$$

where $S[\phi]$ is the integral operator

$$S[\phi(x,h)] = \int\tilde{S}(x-x',h-h')\phi(x',h')\,\mathrm{d}x'\,\mathrm{d}h' \tag{2.13}$$

with the kernel

$$\tilde{S}(x-x',h-h') = \frac{1}{4\pi}\int|k|\exp[\mathrm{i}k(x-x')]\exp[-|k||h-h'|]\,\mathrm{d}k, \tag{2.14}$$

$V(h)$ being the velocity of the unperturbed flow. Substituting $H = H_2$ in (2.9), we obtain the linearized system of equations. The next-order term H_3, which is responsible for three-wave interactions, has the form

$$H_3 = \int \left\{ -\phi_x \eta P[\phi_x] + \tfrac{1}{2}(\nu(h)\eta^2 \phi_x + \nu_h \eta^2 P[\phi_x]) - \tfrac{1}{3}\nu\nu_h\eta^3 \right\} \mathrm{d}x\, \mathrm{d}h, \tag{2.15}$$

where the integral operator

$$P[\phi(x,h)] = \int \tilde{P}(x-x', h-h')\phi(x',h')\,\mathrm{d}x'\,\mathrm{d}h' \tag{2.16}$$

has the kernel

$$\tilde{P}(x-x', h-h') = \frac{1}{4\pi}\int \mathrm{sign}(h-h')\exp[\mathrm{i}k(x-x')]\exp(-|k||h-h'|)\,\mathrm{d}k. \tag{2.17}$$

In terms of a Fourier transform, the governing equations (2.9) have the form

$$\dot{\phi}(k,t,h) = -\frac{\delta H}{\delta\eta^*(k,t,h)} - \mathrm{i}\frac{\nu_h}{k}\frac{\delta H}{\delta\phi^*(k,t,h)}, \quad \dot{\eta}(k,t,h) = \frac{\delta H}{\delta\phi^*(k,t,h)}, \tag{2.18}$$

where an asterisk denotes complex conjugation.

These Hamiltonian equations are convenient for the study of perturbations of flows with large vertical gradients of vorticity in thin (compared to the characteristic wavelength) fluid layers. However, for the flows with large gradients of velocity, it is preferable to introduce Hamiltonian variables in a different way. Instead of (2.8), let us introduce the new variable Φ,

$$\tilde{\Omega}(x,h,t) = \nu(h) - \frac{\partial\Phi}{\partial x}. \tag{2.19}$$

Substituting (2.19) into (2.6), we obtain the canonical equations

$$\dot{\Phi}(x,t,h) = -\frac{\delta H}{\delta\eta(x,t,h)}, \quad \dot{\eta}(x,t,h) = \frac{\delta H}{\delta\Phi(x,t,h)}. \tag{2.20}$$

Then, the kinetic energy normalized by average density is

$$E = -\frac{1}{2}\int (\nu - \Phi_x)(\nu' - \Phi'_{x'})\mathscr{G}(x-x', s-s')\,\mathrm{d}x'\,\mathrm{d}h'\,\mathrm{d}x\,\mathrm{d}h, \tag{2.21}$$

where $\mathscr{G}$ is defined by (2.11), and $\nu = \nu(x,h)$, $\nu' = \nu(x',h')$, $\Phi' = \Phi(x',h')$. Expansion of the Hamiltonian in powers of η and Φ has the form $H = H_2 + H_3 + \ldots$, where

$$H_2 = \frac{1}{2}\int \{\Phi S[\Phi] + 2V(h)\eta\Phi_x - 2\nu\eta P[\Phi_x] - \nu\eta S[\nu\eta] + N^2\eta^2\}\,\mathrm{d}x\,\mathrm{d}h \tag{2.22}$$

and

$$H_3 = \frac{1}{2}\int \{2\nu\eta S[\Phi_x\eta] - 2\eta\Phi_x P[\Phi_x] + \nu\eta^2 P[\nu\eta_{xx}] - \nu\eta^2 S[\Phi_x]\}\,\mathrm{d}x\,\mathrm{d}h, \tag{2.23}$$

and the integral operators $P[f]$ and $S[f]$ are defined by (2.13), (2.14), (2.16), (2.17).

In terms of the Fourier transform with respect to x system (2.20) has the form

$$\dot{\Phi}(k,t,h) = -\frac{\delta H}{\delta\eta^*(k,t,h)}, \quad \dot{\eta}(k,t,h) = \frac{\delta H}{\delta\Phi^*(k,t,h)}. \tag{2.24}$$

In subsequent sections, these Hamiltonian equations will be used for the study of resonant interactions involving a weakly unstable mode.

3. Resonant triplet in the three-layer model involving weakly unstable mode

3.1. *Hamiltonian equations for models with constant vorticity and density in each layer*

System (2.9), and its Fourier-space counterpart (2.18), can be applied to the investigation of perturbation dynamics both in continuously stratified and layered flows. Suppose that the gradients of the unperturbed vorticity and density are large in two layers and equal to zero elsewhere. We assume that the thickness of these two layers is negligible. This assumption, which is justified if the wavelengths at the interfaces are much larger than the thickness of the shear layers, corresponds to the parameters of the stratified flow in the form

$$N^2 = \sum_{j=1}^{2} N_j^2 \delta(h - h_j), \quad \nu_h = \sum_{j=1}^{2} \nu_j \delta(h - h_j). \tag{3.1}$$

Here indices 1, 2 correspond to the lower and upper interfaces respectively. The dependent variables $\phi_j(x, t)$ and $\eta_j(x, t)$ are defined by the equations

$$\phi(h, x, t) = \sum_{j=1}^{2} \phi_j(x, t)\delta(h - h_j), \quad \eta_j(x, t) = \eta(h_j, x, t). \tag{3.2}$$

Physically, ϕ_j represent the difference between hydrodynamical potentials of disturbance velocities on both sides of the interfaces.

Substitution of singular vertical profiles of vorticity and of the Brunt–Väisälä frequency into the equations for ϕ and η gives the equations for layered stratified flow in the form (Goncharov 1986; Goncharov & Pavlov 1993)

$$\frac{\partial}{\partial x}\dot{\phi}_j(x, t) = -\frac{\partial}{\partial x}\frac{\delta H}{\delta \eta_j(x, t)} + \nu_j \frac{\delta H}{\delta \phi_j(x, t)}, \quad \dot{\eta}_j(x, t) = \frac{\delta H}{\delta \phi_j(x, t)}, \tag{3.3}$$

or, in terms of a fourier transform,

$$\dot{\phi}_j(k, t) = -\frac{\delta H}{\delta \eta_j^*(k, t)} - \frac{\mathrm{i}\nu_j}{k}\frac{\delta H}{\delta \phi_j^*(k, t)}, \quad \dot{\eta}_j(k, t) = \frac{\delta H}{\delta \phi_j^*(k, t)}. \tag{3.4}$$

These equations are valid for the perturbation dynamics in an n-layer model with arbitrary n. However, in what follows we consider the case $n = 2$ (three-layer model).

In matrix form, equations (3.4) for $n = 2$ are equivalent to

$$\mathscr{J}(k)\dot{\boldsymbol{d}}(k, t) = -\frac{\delta H}{\delta \boldsymbol{d}(-k, t)}, \tag{3.5}$$

where

$$\boldsymbol{d} = (\phi_1, \phi_2, \eta_1, \eta_2), \quad \boldsymbol{d}^*(k) = \boldsymbol{d}(-k),$$

and

$$\mathscr{J}(k) = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & \mathrm{i}\nu_1/k & 0 \\ 0 & 1 & 0 & \mathrm{i}\nu_2/k \end{pmatrix}$$

Equation (3.5) describes the dynamics of vorticity at the interfaces.

Substitution of (3.1), (3.2) into the Fourier transform of (2.12) gives, for $n = 2$, the expression for the leading-order term of the Hamiltonian,

$$H_2 = \frac{1}{2}\int \sum_{j=1}^{2}\left\{ \frac{|k|}{2}\phi_j^*(k)\phi_j(k) + \frac{|k|}{2}\phi_j^*(k)s_j(k) + 2\mathrm{i}kV_j\eta_j^*(k)\phi_j(k) + \left(N_j^2 - V_j\nu_j\right)\eta_j^*(k)\eta_j(k)\right\}\mathrm{d}k \quad (3.6)$$

where $s_{1,2}(k) = \mu(k)\phi_{2,1}(k)$, $\mu(k) = \exp(-|k||h_1 - h_2|)$, and $V_j = V(h_j)$ is the velocity of the unperturbed flow at the jth interface. The vorticity jump at the jth interface ν_j is $V'(h_j + \epsilon) - V'(h_j - \epsilon)$, where ϵ is an arbitrary small number.

In matrix form, H_2 can be written as

$$H_2 = \frac{1}{2}\int (\boldsymbol{d}^*(k,t), \hat{\boldsymbol{h}}(k)\boldsymbol{d}(k,t))\,\mathrm{d}k,$$

where the parentheses (,) denote the dot product, and

$$\hat{\boldsymbol{h}}(k) = \begin{pmatrix} |k|/2 & \mu(k)|k|/2 & -\mathrm{i}kV_1 & 0 \\ \mu(k)|k|/2 & |k|/2 & 0 & -\mathrm{i}kV_2 \\ \mathrm{i}kV_1 & 0 & \left(N_1^2 - V_1\nu_1\right) & 0 \\ 0 & \mathrm{i}kV_2 & 0 & \left(N_2^2 - V_2\nu_2\right) \end{pmatrix}$$

Here the matrices $\mathscr{J}(k)$ and $\hat{\boldsymbol{h}}(k)$ have symmetry properties (Romanova 1994)

$$\mathscr{J}^*(k) = \mathscr{J}(-k), \quad \mathscr{J}^*(k) = -\mathscr{J}^T(k),$$
$$\hat{\boldsymbol{h}}^*(k) = \hat{\boldsymbol{h}}(-k), \quad \hat{\boldsymbol{h}}^*(k) = \hat{\boldsymbol{h}}^T(k),$$

where the superscript T denotes matrix transposition, and the asterisk complex conjugation. In the next order in nonlinearity, we obtain the expression for H_3,

$$H_3 = \frac{1}{\sqrt{2\pi}}\int \sum_{j=1}^{2}\left\{ \left[-\mathrm{i}k_1\phi_j(k_1) + \tfrac{1}{2}\nu_j\eta_j(k_1)\right]\eta_j(k_2)p_j(k_3) + V_{0j}'\left[\tfrac{1}{2}\mathrm{i}k_1\phi_j(k_1) - \frac{1}{3}\nu_j\eta_j(k_1)\right]\eta_j(k_2)\eta_j(k_3)\right\}\delta(k_1+k_2+k_3)\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3, \quad (3.7)$$

where

$$p_1(k) = -\tfrac{1}{2}\mathrm{i}k\mu(k)\phi_2(k), \quad p_2(k) = \tfrac{1}{2}\mathrm{i}k\mu(k)\phi_1(k),$$

and

$$V_{0j}' = \tfrac{1}{2}[V'(h_j + \epsilon) + V'(h_j - \epsilon)].$$

3.2. *Linearized problem*

Now, the system should be rewritten in normal variables, which diagonalize the linear part of the Hamiltonian. This procedure requires the solution of the linearized problem, obtained by the substitution of the quadratic form H_2 instead of H. As we will see, in the presence of linear instability the exact diagonalization of the Hamiltonian at the linear order is impossible. Instead, the Hamiltonian is reduced to the quasi-diagonal form, with small off-diagonal terms that disappear in the zero instability limit.

Using $H = H_2$ in (3.6), it is easy to obtain from (3.3) the linear system

$$\left.\begin{aligned}\dot{\phi}_1 + \mathrm{i}kV_1\phi_1 &= -N_1^2\eta_1 - \frac{\mathrm{i}\nu_1}{2}\operatorname{sign} k(\phi_1 + \mu\phi_2),\\ \dot{\phi}_2 + \mathrm{i}kV_2\phi_2 &= -N_2^2\eta_2 - \frac{\mathrm{i}\nu_2}{2}\operatorname{sign} k(\phi_2 + \mu\phi_1),\\ \dot{\eta}_1 + \mathrm{i}kV_1\eta_1 &= \frac{|k|}{2}(\phi_1 + \mu\phi_2),\\ \dot{\eta}_2 + \mathrm{i}kV_2\eta_2 &= \frac{|k|}{2}(\phi_2 + \mu\phi_1).\end{aligned}\right\} \tag{3.8}$$

Then, dispersion relation is

$$\det(\hat{\boldsymbol{h}}(k) - \mathrm{i}\omega\mathscr{J}(k)) = 0$$

or, alternatively,

$$\mathscr{D}(\omega, k) \equiv D_1(\omega, k)D_2(\omega, k) - \mu^2(k)k^2/4 = 0, \tag{3.9}$$

where

$$D_j = \Omega_j/b_j - |k|/2, \quad \Omega_j = \omega - kV_j, \quad b_j = N_j^2/\Omega_j + \nu_j/k. \tag{3.10}$$

Here ω is frequency, and $N_j^2 = g(\rho_j - \rho_{j+1})/\rho_0$. Dispersion equations $D_{1,2}(\omega, k) = 0$ describe uncoupled linear waves at the interfaces.

Eigenvectors of the linear problem $\boldsymbol{Z} = (\phi_{01}, \phi_{02}, \eta_{01}, \eta_{02})$ are obtained from the equation

$$(\hat{\boldsymbol{h}}(k) - \mathrm{i}\omega\mathscr{J}(k))\boldsymbol{Z} = 0,$$

and we choose the normalization condition

$$(\boldsymbol{Z}^*, \mathscr{J}\boldsymbol{Z}) = -\mathrm{i}. \tag{3.11}$$

Then, the components of $\boldsymbol{Z}$ are

$$\begin{aligned}\phi_{01} &= (D_2/L)^{1/2}, & \phi_{02} &= -(D_1/L)^{1/2},\\ \eta_{01} &= \frac{\mathrm{i}}{b_1}(D_2/L)^{1/2}, & \eta_{02} &= -\frac{\mathrm{i}}{b_2}(D_1/L)^{1/2},\end{aligned} \tag{3.12}$$

where

$$L = \mathscr{D}_\omega(\omega, k) = D_{1\omega}D_2 + D_{2\omega}D_1. \tag{3.13}$$

As is well known (cf. Whitham 1974, Chapter 11), the sign of L defines the sign of wave action density in a wave system. We choose the branches of the dispersion curves for which the sign of L is positive, and then the sign of the energy of a normal mode is defined by the sign of the frequency. Due to normalization condition (3.11), the expression for energy density takes the form ωaa^*.

3.3. *Resonant triplet for the three-layer model*

We consider the resonant interaction of three spectrally narrow wave packets, when two packets are stable and the third one is weakly unstable, the instability being caused by the linear coupling of modes. Two modes are coupled if the branches of their dispersion curves, corresponding to different signs of energy, intersect for $\mu(k) = 0$ at a certain point (cf. Craik 1985, Chapter 2). Linear instability caused by the coupling is weak, provided that the right-hand side of the dispersion equation is small, $\mu(k)k/2 \ll 1$. However, in a certain neighbourhood of the intersection point

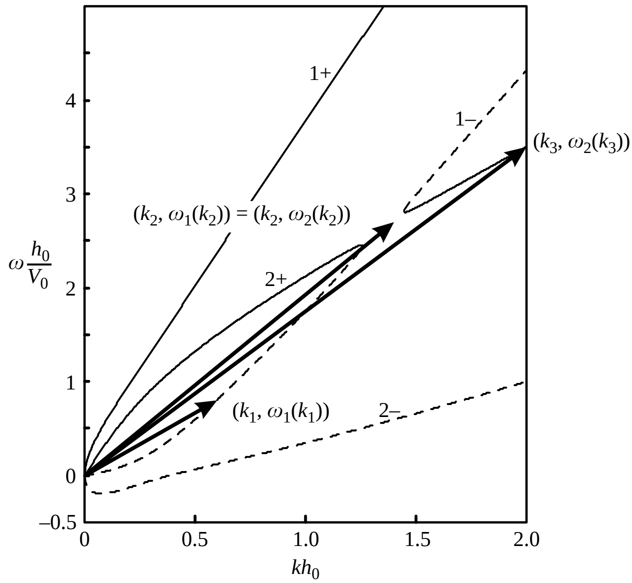

FIGURE 1. Plot of the dispersion relation (3.14) and the resonant triplet involving the weakly unstable wave, for $R_1 = 2.0$, $R_2 = 1.5$. Solid and dashed curves correspond to positive and negative signs of wave action, respectively.

even the small coupling is significant. It is important to note that the intersection of dispersion curves with the same sign of energy leads not to instability, but to change of identity of modes (Craik 1985; Romanova 1998). Below we will assume that the signs of energy are different, i.e. coupling leads to instability.

First, we will demonstrate the possibility of resonant interactions involving weakly unstable waves in the three-layer model. Let us consider a velocity profile of the form $V(z) = 3V_0$ for $z < -h_0$, $V(z) = 2V_0 - zV_0/h_0$ for $-h_0 < z < h_0$, and $V(z) = V_0$ for $z > h_0$. Here $2h_0$ is the distance between interfaces.

Then $\nu_1 = V_0/h_0$, $\nu_2 = -V_0/h_0$, and the dispersion relation (3.9) can be written in the dimensionless form

$$\left(1 - \frac{2(\tilde{\omega} - 3\tilde{k})^2}{R_1\tilde{k} - (\tilde{\omega} - 3\tilde{k})}\right)\left(1 - \frac{2(\tilde{\omega} - \tilde{k})^2}{R_2\tilde{k} + (\tilde{\omega} - \tilde{k})}\right) = \exp(-4\tilde{k}) \tag{3.14}$$

where $\tilde{k} = kh_0$, $\tilde{\omega} = \omega h_0/V_0$, and non-dimensional variables $R_j = N_j^2 h_0/V_0^2$ have the sense of local Richardson numbers. Dispersion curves for (3.14) are shown in figure 1. Note that in this example $\tilde{k} = kh_0 \sim 1.5$, and the right-hand side of (3.14) is small in the region of coupling.

Two pairs of modes described by the approximate equations $D_1(\omega, k) \approx 0$, $D_2(\omega, k) \approx 0$ are labelled in figure 1 as $1^{(\pm)}$ and $2^{(\pm)}$ respectively. For sufficiently large kh_0, the former pair of modes have maxima at the lower interface, and the latter at the upper one. Signs + and − correspond to the sign of the wave action density, defined as the sign of the expression $\partial D_{1,2}(\omega, k)/\partial\omega$ if the interfaces are sufficiently distant.

Figure 1 demonstrates the existence of resonant triplets with the required properties. However, it makes sense to present the same dispersion curves and the resonant triplet

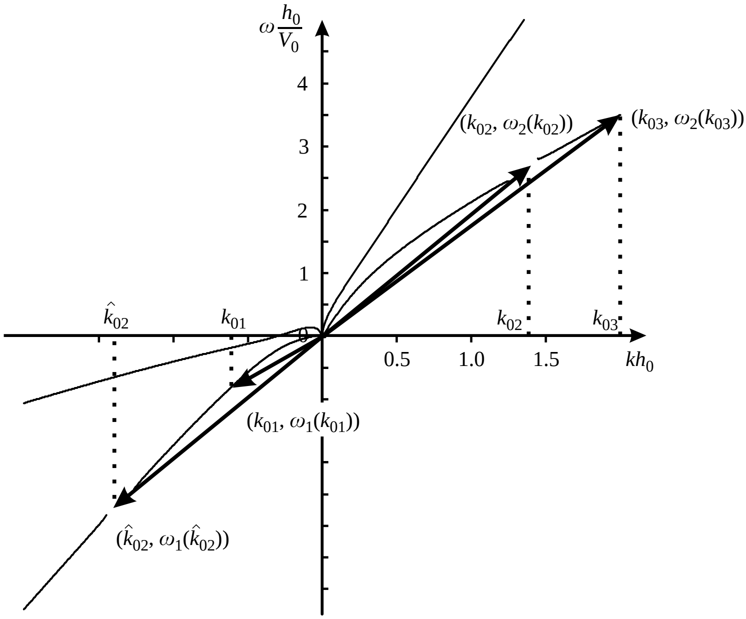

FIGURE 2. Dispersion curve of the three-layer model where all the branches correspond to positive sign of the wave action. The modes shown satisfy simultaneously both resonance conditions (3.15), (3.16).

in a different form, using the fact that the pairs (ω, k) and $(-\omega, -k)$ correspond to the same physical state, but to different signs of wave action density. This allows the dispersion curves in figure 2 to be redrawn, leaving only the branches that correspond to positive sign of wave action density, as discussed above.

Figure 2 demonstrates that we have two resonant triplets of the form

$$k_{01} - k_{02} + k_{03} = 0, \quad \omega_1(k_{01}) - \omega_2(k_{02}) + \omega_2(k_{03}) = 0, \tag{3.15}$$

$$k_{01} + \hat{k}_{02} + k_{03} = 0, \quad \omega_1(k_{01}) + \omega_1(\hat{k}_{02}) + \omega_2(k_{03}) = 0, \tag{3.16}$$

where $\hat{k}_{02} = -k_{02}$. Both resonant conditions are satisfied simultaneously, since at the midpoint of the instability interval

$$\omega_2(k_{02}) = -\omega_1(\hat{k}_{02}) = -\omega_1(-k_{02}). \tag{3.17}$$

However, as shown in the Appendix, the coefficient of the explosive resonant interaction corresponding to (3.16) is small, so that the nonlinear terms corresponding to this interaction can be neglected.

In what follows we shall consider spectrally narrow wave packets centred at the points k_{01}, k_{02}, k_{03}. The long-wave packet centred at k_{01} interacts with two short-wave packets. The packet of short unstable waves corresponding to the second mode is centred at the midpoint of the narrow interval of instability k_{02}, and the packet of short unstable waves corresponding to the first mode is centred at the point $\hat{k}_{02}$. The packet of short stable waves corresponding to the second mode is centred at the point k_{03} (see figure 2).

3.4. *Generalized normal variables*

Consider first the well-studied case of neutrally stable waves. Following Zakharov (1974), we perform the transformation to normal variables $a_{1,2}(k, t)$ using the formula

$$\boldsymbol{d}(k, t) = \sum_{j=1}^{2} [\boldsymbol{Z}_j(k) a_j(k, t) + \boldsymbol{Z}_j^*(-k) a_j^*(-k, t)], \tag{3.18}$$

where $\boldsymbol{Z}_j(k)$ are normalized eigenvectors (3.12) corresponding to eigenfrequencies $\omega_j(k)$. It can be shown that $\boldsymbol{Z}_j^*(-k)$ is also an eigenvector of the linearized system. In a more compact form, transformation (3.18) can be written as

$$\boldsymbol{d}(k, t) = \mathscr{Z}(k) \boldsymbol{a}(k, t),$$

where the columns of the matrix $\mathscr{Z}(k)$ are the eigenvectors $\boldsymbol{Z}_j(k)$, $\boldsymbol{Z}_j^*(-k)$, and the vector of new variables $\boldsymbol{a}(k, t)$ is

$$\boldsymbol{a}(k, t) = (a_1(k, t), a_2(k, t), a_1^*(-k, t), a_2^*(-k, t)).$$

In terms of $\boldsymbol{a}(k, t)$, the dynamical system has the same form as (3.5),

$$\tilde{\mathscr{J}}(k) \dot{\boldsymbol{a}}(k, t) = -\frac{\delta H}{\delta \boldsymbol{a}(-k, t)}, \tag{3.19}$$

where

$$\tilde{\mathscr{J}}(k) = (\mathscr{Z}^T(-k),\ \mathscr{J}(k) \mathscr{Z}(k)). \tag{3.20}$$

Due to the normalization condition (3.11) and the orthogonality of eigenvectors corresponding to different real eigenvalues (Romanova 1998), the transformed matrix $\tilde{\mathscr{J}}(k)$ has the canonical form

$$\tilde{\mathscr{J}}(k) = \begin{pmatrix} 0 & 0 & \mathrm{i} & 0 \\ 0 & 0 & 0 & \mathrm{i} \\ -\mathrm{i} & 0 & 0 & 0 \\ 0 & -\mathrm{i} & 0 & 0 \end{pmatrix}. \tag{3.21}$$

Thus, the dynamical equations in terms of $a_j(k, t)$ have the well-known canonical form (Zakharov 1974; Kuznetsov & Zakharov 1997)

$$\dot{a}_j(k, t) = -\mathrm{i} \frac{\delta H}{\delta a_j^*(k, t)}, \quad j = 1, 2. \tag{3.22}$$

The quadratic term in the expansion of the Hamiltonian H_2 is

$$H_2 = \frac{1}{2} \int (\boldsymbol{a}(-k), \tilde{\boldsymbol{h}}(k) \boldsymbol{a}(k))\, \mathrm{d}k, \tag{3.23}$$

where the matrix $\tilde{\boldsymbol{h}}(k)$, which defines the structure of H_2, has the form

$$\tilde{\boldsymbol{h}}(k) = (\mathscr{Z}^T(-k), \hat{\boldsymbol{h}}(k) \mathscr{Z}(k)), \tag{3.24}$$

and it is easy to show that H_2 has the diagonal form

$$H_2 = \int \sum_{j=1}^{2} \omega_j a_j a_j^*\, \mathrm{d}k. \tag{3.25}$$

However, in the region of coupling this transformation cannot be used, since L, which appears in the denominators of (3.12), is small within this region and becomes zero at certain points. Due to this, in this region we perform the transformation to variables $a_{1,2}$ according to the same formula (3.18), but now $\boldsymbol{Z}_j$ are the eigenvectors of the uncoupled linearized problem, for $\mu(k) = 0$ (see Romanova 1998). In this case $D_1(\omega_1(-k_{02}), -k_{02}) = D_2(\omega_2(k_{02}), k_{02}) = 0$, and equations (3.11), (3.12), (3.13) give expressions for the eigenvectors in the form

$$\boldsymbol{Z}_1^*(-k_{02}) = \frac{1}{\sqrt{D_{1\omega}}}(1, 0, \mathrm{i}/b_1(k_{02}), 0), \tag{3.26}$$

$$\boldsymbol{Z}_2(k_{02}) = \frac{1}{\sqrt{D_{2\omega}}}(0, -1, 0, -\mathrm{i}/b_2(k_{02})), \tag{3.27}$$

where $b_j(k)$ are given in (3.10). Then, if the condition (3.17) is satisfied, two independent and orthogonal eigenvectors $\boldsymbol{Z}_1^*(-k_{02})$ and $\boldsymbol{Z}_2(k_{02})$ correspond to the same eigenvalue $\omega_2(k_{02}) = -\omega_1(-k_{02})$.

Let us now use eigenvectors (3.26), (3.27) in (3.18). The transformed matrix $\tilde{\mathscr{J}}(k)$ has the same form (3.21), so the dynamical equations are again (3.22). On the other hand, the matrix $\tilde{\boldsymbol{h}}(k)$, which determines the structure of the quadratic Hamiltonian H_2 in terms of $a_j(k,t)$, is

$$\tilde{\boldsymbol{h}}(k) = \begin{pmatrix} 0 & f_1(k) & \omega_1(-k) & f_2(-k) \\ f_1(-k) & 0 & f_2(-k) & \omega_2(-k) \\ \omega_1(k) & f_2(k) & 0 & f_1(-k) \\ f_2(k) & \omega_2(k) & f_1(k) & 0 \end{pmatrix},$$

where

$$f_1(k) = \frac{\mu(k)|k|}{2\sqrt{D_{1\omega}(k)D_{2\omega}(-k)}}, \qquad f_2(k) = \frac{\mu(k)|k|}{2\sqrt{D_{1\omega}(k)D_{2\omega}(k)}}.$$

Note that $f_1(k)$ and $f_2(k)$ are proportional to the coupling parameter $\mu(k)$. Substituting $\tilde{\boldsymbol{h}}(k)$ into (3.23), we obtain the expression for H_2 in the region of coupling,

$$H_2 = \int \left\{ \sum_{j=1}^{2} \omega_j a_j a_j^* + [f_1(k)a_1(-k)a_2(k) + f_2(k)a_1(k)a_2^*(k) + \text{c.c.}] \right\} \mathrm{d}k. \tag{3.28}$$

Strictly speaking, the variables a_1, a_2 are not normal variables in the classical sense, and the quadratic Hamiltonian H_2 is not reduced to the diagonal form. There are small additional terms, responsible for the linear instability. Both variables are the combinations of the growing and decaying modes in the instability region.

A derivation of the expression for the cubic term H_3 in terms of $a_j(k)$ is presented in the Appendix. As discussed above, the term corresponding to the explosive interaction is small and can be neglected, so that the essential term of H_3 has the form

$$H_3 = \frac{1}{\sqrt{2\pi}} \int [\mathscr{B}_2(k_1, k_2, k_3)a_1(k_1)a_2^*(k_2)a_2(k_3)\delta(k_1 - k_2 + k_3) + \text{c.c.}]\, \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3. \tag{3.29}$$

In the Appendix, the expression for $\mathscr{B}_2$, in particular at the points (k_{01}, k_{02}, k_{03}), is derived.

3.5. *Evolution equations*

Provided that resonance conditions (3.15), (3.16) are satisfied, let us make the substitution

$$a_1(k) = c_1(k_{01} + \kappa) + c_4(-(k_{02} + \kappa)), \qquad a_2(k) = c_2(k_{02} + \kappa) + c_3(k_{03} + \kappa), \tag{3.30}$$

where $\kappa \ll k_j$.

Since $\mathscr{B}_2$ is a continuous function of its arguments, it can be taken out of the integrand, so that the cubic Hamiltonian H_3 takes the form

$$H_3 = \frac{1}{\sqrt{2\pi}} W(k_{01}, k_{02}, k_{03}) \int c_1(k_{01}+\kappa_1) c_2^*(k_{02}+\kappa_2) c_3(k_{03}+\kappa_3) \times \delta(\kappa_1 - \kappa_2 + \kappa_3)\, \mathrm{d}\kappa_1\, \mathrm{d}\kappa_2\, \mathrm{d}\kappa_3 + \text{c.c.}, \qquad (3.31)$$

where $W(k_{01}, k_{02}, k_{03}) = 2\mathscr{B}_2(k_{01}, k_{02}, k_{03})$.

Evolution equations governing the process of resonant interaction are derived using (3.22) with $H = H_2 + H_3$. Here, the leading-order term of the Hamiltonian H_2 has the form (3.25) in the vicinity of the points k_{01}, k_{03}, and the form (3.28) in the vicinity of k_{02}, $\hat{k}_{02}$.

If we consider unstable linear coupling of modes, so that $-\omega_1(-k) = \omega_2(k)$, the term $f_2(k)a_1(k)a_2^*(k)$ in the integrand of (3.28) is oscillating quickly and can be neglected. However, if the coupled modes have the same sign as the wave action, corresponding to the condition $\omega_1(k) = \omega_2(k)$, this term is resonant and should be taken into account, while the first term $f_1(k)a_1(-k)a_2(k)$ can be omitted. As it was mentioned above, in the latter case the coupling of waves leads to the change of identity of modes.

Thus, from (3.22), (3.25), (3.28) and (3.31) we obtain the system of evolution equations

$$\dot{c}_1(k_{01}+\kappa) = -\mathrm{i}\frac{\delta(H_2+H_3)}{\delta c_1^*(k_{01}+\kappa)} = -\mathrm{i}\omega_1(k_{01}+\kappa)c_1(k_{01}+\kappa) - \frac{\mathrm{i}}{\sqrt{2\pi}} W^*(k_{01}, k_{02}, k_{03}) \int c_2(k_{02}+\kappa_2) c_3^*(k_{03}+\kappa_3) \delta(\kappa - \kappa_2 + \kappa_3)\, \mathrm{d}\kappa_2\, \mathrm{d}\kappa_3, \qquad (3.32a)$$

$$\dot{c}_3(k_{03}+\kappa) = -\mathrm{i}\frac{\delta(H_2+H_3)}{\delta c_3^*(k_{03}+\kappa)} = -\mathrm{i}\omega_2(k_{03}+\kappa)c_3(k_{03}+\kappa) - \frac{\mathrm{i}}{\sqrt{2\pi}} W^*(k_{01}, k_{02}, k_{03}) \int c_1^*(k_{01}+\kappa_1) c_2(k_{02}+\kappa_2) \delta(\kappa_1 - \kappa_2 + \kappa)\, \mathrm{d}\kappa_1\, \mathrm{d}\kappa_2, \qquad (3.32b)$$

$$\dot{c}_2(k_{02}+\kappa) = -\mathrm{i}\frac{\delta(H_2+H_3)}{\delta c_2^*(k_{02}+\kappa)} = -\mathrm{i}\omega_2(k_{02}+\kappa)c_2(k_{02}+\kappa) - \mathrm{i} f_1(k_{02}) c_4^*(-(k_{02}+\kappa)) - \frac{\mathrm{i}}{\sqrt{2\pi}} W(k_{01}, k_{02}, k_{03}) \int c_1(k_{01}+\kappa_1) c_3(k_{03}+\kappa_3) \delta(\kappa_1 - \kappa + \kappa_3)\, \mathrm{d}\kappa_1\, \mathrm{d}\kappa_3, \qquad (3.32c)$$

$$\dot{c}_4(-k_{02}-\kappa) = -\mathrm{i}\frac{\delta(H_2+H_3)}{\delta c_4^*(-k_{02}-\kappa)} = -\mathrm{i}\omega_1(-k_{02}-\kappa)c_4(-k_{02}-\kappa) - \mathrm{i} f_1(k_{02}) c_2^*(k_{02}+\kappa). \qquad (3.32d)$$

Taking the complex conjugate of equation (3.32d),

$$\dot{c}_4^*(-k_{02}-\kappa) = \mathrm{i}\omega_1(-k_{02}-\kappa)c_4^*(-k_{02}-\kappa) + \mathrm{i} f_1(k_{02}) c_2(k_{02}+\kappa).$$

As discussed above, in H_2 in the vicinity of the point k_{02} only the term leading to linear resonance is kept. In (3.32d), the nonlinear term corresponding to explosive interaction is omitted, since, as shown in the Appendix, it is of higher order than the nonlinear terms in other equations.

In order to study the slow evolution of amplitudes of wave packets in time and space, let us introduce the variables A_j, $j = 1, 2, 3, 4$ as

$$\left.\begin{aligned}
c_1(k_{01}+\kappa, t) &= \varepsilon \exp(-\mathrm{i}\omega_1(k_{01})t)A_1(\kappa, T),\\
c_2(k_{02}+\kappa, t) &= \varepsilon \exp(-\mathrm{i}\omega_2(k_{02})t)A_2(\kappa, T),\\
c_3(k_{03}+\kappa, t) &= \varepsilon \exp(-\mathrm{i}\omega_2(k_{03})t)A_3(\kappa, T),\\
c_4^*(-k_{02}-\kappa, t) &= \varepsilon \exp(\mathrm{i}\omega_1(-k_{02})t)A_4(\kappa, T)\\
&= \varepsilon \exp(-\mathrm{i}\omega_2(k_{02})t)A_4(\kappa, T).
\end{aligned}\right\} \tag{3.33}$$

Here, ε is the small parameter of nonlinearity, and the amplitudes A_j are assumed to have the order of unity and depend on the 'slow' time $T = \varepsilon t$. The wavepackets are narrow, with spectral width $\kappa/k_{0j} \sim \varepsilon$, and we also assume that ε and the coupling parameter $\mu(k)$ are of the same order. Let us expand

$$\omega_j(k_0+\kappa) = \omega_j(k_0) + v_j(k_0)\kappa, \quad v_j(k_0) = \left.\frac{\mathrm{d}\omega_j}{\mathrm{d}k}\right|_{k=k_0}.$$

In terms of A_j, the system of evolution equations is

$$\begin{aligned}
&\frac{\partial A_1(\kappa, T)}{\partial T} + \frac{\mathrm{i}\kappa}{\varepsilon} v_1(k_{01})A_1(\kappa, T)\\
&\quad = -\frac{\mathrm{i}}{\sqrt{2\pi}} W^* \int A_2(\kappa_2, T)A_3^*(\kappa_3, T)\delta(\kappa - \kappa_2 + \kappa_3)\,\mathrm{d}\kappa_2\,\mathrm{d}\kappa_3,\\
&\frac{\partial A_3(\kappa, T)}{\partial T} + \frac{\mathrm{i}\kappa}{\varepsilon} v_2(k_{03})A_3(\kappa, T)\\
&\quad = -\frac{\mathrm{i}}{\sqrt{2\pi}} W^* \int A_1^*(\kappa_1, T)A_2(\kappa_2, T)\delta(\kappa_1 - \kappa_2 + \kappa)\,\mathrm{d}\kappa_1\,\mathrm{d}\kappa_2,\\
&\frac{\partial A_2(\kappa, T)}{\partial T} + \frac{\mathrm{i}\kappa}{\varepsilon} v_2(k_{02})A_2(\kappa, T)\\
&\quad = -\mathrm{i}sA_4(\kappa, T) - \frac{\mathrm{i}}{\sqrt{2\pi}} W \int A_1(\kappa_1, T)A_3(\kappa_3, T)\delta(\kappa_1 - \kappa + \kappa_3)\,\mathrm{d}\kappa_1\,\mathrm{d}\kappa_3,\\
&\frac{\partial A_4(\kappa, T)}{\partial T} + \frac{\mathrm{i}\kappa}{\varepsilon} v_1(-k_{02})A_4(\kappa, T) = \mathrm{i}sA_2(\kappa, T),
\end{aligned}$$

where $s = f_1(k_{02})/\varepsilon$. Taking the inverse Fourier transform with respect to κ,

$$B_j(X, T) = \frac{1}{\sqrt{2\pi}} \int A_j(\kappa, T)\exp(\mathrm{i}\kappa x)\,\mathrm{d}\kappa,$$

where $j = 1, 2, 3, 4$, and using the relation

$$\delta(\kappa_1 - \kappa + \kappa_3) = \frac{1}{2\pi} \int \exp(\mathrm{i}(\kappa_1 - \kappa + \kappa_3)x')\,\mathrm{d}x',$$

we obtain, finally,

$$\left.\begin{aligned}
&\frac{\partial B_1(X,T)}{\partial T}+v_1\frac{\partial B_1(X,T)}{\partial X}+\mathrm{i}W^*B_2(X,T)B_3^*(X,T)=0,\\
&\frac{\partial B_3(X,T)}{\partial T}+v_3\frac{\partial B_3(X,T)}{\partial X}+\mathrm{i}W^*B_1^*(X,T)B_2(X,T)=0,\\
&\frac{\partial B_2(X,T)}{\partial T}+v_2\frac{\partial B_2(X,T)}{\partial X}+\mathrm{i}sB_4(X,T)+\mathrm{i}WB_1(X,T)B_3(X,T)=0,\\
&\frac{\partial B_4(X,T)}{\partial T}+\hat{v}_2\frac{\partial B_4(X,T)}{\partial X}-\mathrm{i}sB_2(X,T)=0.
\end{aligned}\right\} \tag{3.34}$$

Here $v_1 = v_1(k_{01})$, $v_3 = v_2(k_{03})$, $v_2 = v_2(k_{02})$, $\hat{v}_2 = v_1(-k_{02})$, and $X = \varepsilon x$. For three harmonic waves instead of wave packets, the amplitudes are governed by an ODE system

$$\left.\begin{aligned}
&\dot{B}_1+\mathrm{i}W^*B_2B_3^*=0,\\
&\dot{B}_3+\mathrm{i}W^*B_2B_1^*=0,\\
&\ddot{B}_2-s^2B_2+W^2(|B_1|^2+|B_3|^2)B_2=0.
\end{aligned}\right\} \tag{3.35}$$

This system is the particular case of the system obtained by Loesch (1974) and Pedlosky (1975) that described the resonant triad of inviscid baroclinic waves for the case when one of the three waves is marginally unstable. As shown analytically by Pedlosky (1975), this system possesses a periodical solution. This means that the process of nonlinear interacion of the linearly unstable wave with neutral ones can stabilize its growth.

The method used to obtain the equations written above is universal and does not depend on the physical nature of the coupled waves. All the specific nature of a particular problem is in the coefficients. So, these equations are universal, and can be used for investigation of the nonlinear dynamics for weakly coupled waves of any nature, if we can write the governing system in the Hamiltonian form.

4. Resonant interaction involving weakly unstable mode in the KH model

4.1. *Hamiltonian structure for models of KH type*

In the preceding section, we considered weak linear instability caused by the weak coupling of modes, when there were two distinct modes at the point k, where the eigenfrequencies corresponding to different dispersion branches in the absence of coupling coincide. If the instability takes place in the framework of a single mode, the Hamiltonian structure is different, and the evolution equations that describe the three-wave interaction with the participation of the weakly unstable wave packet have a different form. This case is considered in the present section, again within the framework of the Boussinesq approximation. To obtain evolution equations in this case, we will use the Hamiltonian equations (2.20) and expression (2.21) for the kinetic energy.

Let us consider the undisturbed state with two fluid layers having constant density and velocity, and a thin shear layer between them. If the thickness of the latter is small compared to the characteristic wavelength, the shear layer can be treated as the interface, so that velocity $V(h)$, density $\rho(h)$ and vorticity $\nu(h)$ of the unperturbed flow can be represented as

$$\left.\begin{aligned} V(h) &= V_2\Theta(h-h_0) + V_1\Theta(h_0-h), \quad \nu(h) = -\Delta V\delta(h-h_0),\\ \rho(h) &= \rho_2\Theta(h-h_0) + \rho_1\Theta(h_0-h), \quad \rho_h(h) = -\Delta\rho\delta(h-h_0), \end{aligned}\right\} \tag{4.1}$$

where $\Delta V = (V_1 - V_2)$, $\Delta\rho = (\rho_1 - \rho_2)$ and $\Theta(h)$ is the Heaviside function,

$$\Theta(h) = 0, \quad h < 0,$$
$$\Theta(h) = 1, \quad h > 0.$$

The horizontal velocity at the interface $h = h_0$ is $v = (V_1 + V_2)/2$ and the density is $\rho = (\rho_1 + \rho_2)/2$. Here $\rho_{1,2}$ and $V_{1,2}$ are densities and velocities in the lower and upper layers, respectively. This two-layer vertical structure of the undisturbed state is usually called the Kelvin–Helmholtz model.

Now we again turn to the canonical system (2.20). Substitution

$$\Phi(x,t,h) = \varphi(x,t)\delta(h-h_0), \quad \eta(x,t) = \eta(x,t,h_0) \tag{4.2}$$

gives the canonical system in the form

$$\dot\varphi(x,t) = -\frac{\delta H}{\delta\eta(x,t)}, \quad \dot\eta(x,t) = \frac{\delta H}{\delta\varphi(x,t)}, \tag{4.3}$$

and, in terms of the Fourier transform,

$$\dot\varphi(k,t) = -\frac{\delta H}{\delta\eta^*(k,t)}, \quad \dot\eta(k,t) = \frac{\delta H}{\delta\varphi^*(k,t)}, \tag{4.4}$$

where $\varphi^*(k,t) = \varphi(-k,t)$, $\eta^*(k,t) = \eta(-k,t)$, since $\varphi(x,t)$ and $\eta(x,t)$ are real.

The same canonical structure was obtained by Benjamin & Bridges (1997) for the more general case of three-dimensional flow, without the Boussinesq approximation, using the dynamical and kinematic conditions at the disturbed interface. Our approach is different, being based on semi-Lagrangian variables (Virasoro 1981; Ripa 1981). This approach allows us, first, to consider perturbations in layered and continuously stratified flows, and, second, to obtain next-order terms in the expansion of the Hamiltonian in a simple way. Generalization of the results to three-dimensional flows is possible, and the Boussinesq approximation can be also dropped.

In vector form, (4.4) is written as

$$\boldsymbol{J}\dot{\boldsymbol{d}}(k,t) = -\frac{\delta H}{\delta\boldsymbol{d}(-k,t)}, \tag{4.5}$$

where the vector $\boldsymbol{d}(k,t)$ is

$$\boldsymbol{d}(k,t) = (\varphi(k,t), \eta(k,t)),$$

and the matrix $\boldsymbol{J}$ has the canonical form

$$\boldsymbol{J} = \begin{pmatrix} 0 & -1\\ 1 & 0 \end{pmatrix}.$$

Substituting (4.1) and (4.2) into (2.22), we obtain the leading-order term of the Hamiltonian in the KH model,

$$H_2 = \frac{1}{2}\int\left(\varphi\bar S[\varphi] - (\Delta V)^2\eta\bar S[\eta] - 2v\eta_x\varphi + g\frac{\Delta\rho}{\rho}\eta^2 - \frac{\sigma\eta_x^2}{\rho}\right)\mathrm{d}x, \tag{4.6}$$

where

$$\bar{S}[f(x)] = \frac{1}{4\pi}\int |k| \exp[ik(x-x')]\, f(x')\, dk dx',$$

and we have also inserted the surface tension term, with the coefficient σ. In terms of a Fourier transform, the Hamiltonian H_2 is

$$H_2 = \frac{1}{2}\int (\boldsymbol{d}^*(k,t), \boldsymbol{h}(k)\boldsymbol{d}(k,t))\, dk, \tag{4.7}$$

where

$$\boldsymbol{h}(k) = \begin{pmatrix} A(k) & -ikv \\ ikv & C(k) \end{pmatrix}$$

and

$$C(k) = g\Delta\rho/\rho + \sigma k^2/\rho - (\Delta V)^2|k|/2, \quad A(k) = |k|/2.$$

In order to obtain the linearized equations, substitute (4.7) into (4.4). The dispersion relation is

$$\det(\boldsymbol{h} - i\omega_{1,2}\boldsymbol{J}) = 0,$$

and the eigenfrequencies are equal to

$$\omega_{1,2} = kv \pm \mathrm{sign}k \left(g\frac{\Delta\rho}{2\rho}|k| + \frac{\sigma}{2\rho}|k|^3 - \frac{(\Delta V)^2 k^2}{4}\right)^{1/2}. \tag{4.8}$$

Instability arises when two eigenfrequencies corresponding to different energy signs coincide, and the expression in brackets in (4.8) vanishes at the wavenumber $k_0 = (g\Delta\rho/\sigma)^{1/2}$ (Drazin 1970; Triantafyllou 1994).

The critical value of the velocity shift ΔV_c for instability is $\Delta V_c = 2(g\sigma\Delta\rho/\rho^2)^{1/4}$. Let us consider the velocity shift ΔV close to this critical value, with the small deviation $\gamma = \Delta V - \Delta V_c$, and wavenumbers close to k_0, $k = k_0 + \kappa$, where κ/k and $\gamma/\Delta V_c$ are small parameters of the same order. Then the frequency values are

$$\omega_{1,2} = \hat{\omega}_0 + v\kappa \pm \mathrm{sign}\, k_0\, \delta, \tag{4.9}$$

where

$$\delta = \left[2(\Delta V_c)^2\kappa^2 - \tfrac{1}{2}\Delta V_c \gamma k_0^2\right]^{1/2}, \quad \hat{\omega}_0 = vk_0,$$

and $\delta/\hat{\omega}_0 \ll 1$. The flow with $\gamma < 0$ is weakly subcritical, and the eigenfrequencies are real. If $\gamma > 0$, the flow is weakly supercritical. Then in the vicinity of the critical wavenumber k_0 the $\omega_{1,2}$ are complex conjugated, and the flow is unstable, with a small growth rate. In fact the flow is also unstable if $\gamma = 0$, but then the instability is of algebraic type.

Now, let us consider a generic Hamiltonian system described by the Hamiltonian equations (4.4), with the first term of the Hamiltonian expansion in the form (4.7). In this general context, we assume that the eigenfrequencies in the dispersion relation coalesce at a certain point k_0, and in the vicinity of this point the dispersion relation has the form (4.9), with

$$\delta = [A(k_0+\kappa)C(k_0+\kappa)]^{1/2} = V_0\left(\kappa^2 + bk_0^2\right)^{1/2},$$

where $A(k) = A(-k) = A^*(k)$ and $C(k) = C(-k) = C^*(k)$. Let us assume that $C(k_0+\kappa)$ (and, hence, δ) is small, while $A(k_0+\kappa)$ is not small. Here V_0 is a parameter with the dimension of velocity, and b is a small non-dimensional parameter of the wave system that can be positive or negative, corresponding to the stable and unstable cases respectively. In the particular case of the KH model, this parameter is the deviation of the velocity shift from its critical value.

4.2. *Generalized normal variables*

Following Romanova (1994), we perform the transformation to new variables $a(k,t)$,

$$\boldsymbol{d}(k,t) = \mathscr{Z}(k)a(k,t) + \mathscr{Z}^*(-k)a^*(-k,t). \tag{4.10}$$

The expression for the vector $\mathscr{Z}$ depends on the value of k. If k lies in the stability region, then

$$\mathscr{Z}(k) = \begin{cases} \mathbf{z}_1(k), & k > 0, \\ \mathbf{z}_2(k), & k < 0. \end{cases} \tag{4.11}$$

Here $\mathbf{z}_{1,2}$ are the eigenvectors defined by the system of linear algebraic equations

$$(\boldsymbol{h} - \mathrm{i}\omega_{1,2}\boldsymbol{J})\mathbf{z}_{1,2} = 0, \tag{4.12}$$

and thus

$$\mathbf{z}_1 = c_1(-\mathrm{i}\ \mathrm{sign}\ k\sqrt{C(k)/A(k)}, 1), \quad \mathbf{z}_2 = c_2(\mathrm{i}\ \mathrm{sign}\ k\sqrt{C(k)/A(k)}, 1), \tag{4.13}$$

where $c_{1,2}$ are arbitrary constants. Using the normalization condition in the form similar to (3.11),

$$(\mathscr{Z}^*(k), \boldsymbol{J}\mathscr{Z}(k)) = -\mathrm{i},$$

we get $c_1 = c_2 = (A(k)/4C(k))^{1/4}$, and then (4.10) gives

$$\varphi(k,t) = \phi_0(k)a(k,t) + \phi_0^*(-k)a^*(-k,t), \tag{4.14a}$$

$$\eta(k,t) = \eta_0(k)a(k,t) + \eta_0^*(-k)a^*(-k,t), \tag{4.14b}$$

where

$$\phi_0(k) = -\mathrm{i}\left(\frac{C(k)}{4A(k)}\right)^{1/4}, \quad \eta_0(k) = \left(\frac{A(k)}{4C(k)}\right)^{1/4}. \tag{4.15}$$

Substitution of (4.14a,b) into (4.4) leads to the well-known canonical system in normal variables (Zakharov 1974):

$$\dot{a}(k,t) = -\mathrm{i}\frac{\delta H}{\delta a^*(k,t)}.$$

The density of the quadratic part of the Hamiltonian H_2 in terms of normal variables is $\omega(k)a(k,t)a^*(k,t)$, where $\omega(k) = vk + \delta$.

However, this transformation fails if k approaches k_0, where $C(k)$ and, hence, δ tends to zero. Then, eigenvalues and eigenvectors tend to each other, $c_{1,2} = \sqrt{A/2\delta}$ approach infinity, and for $\delta = 0$ we get a double eigenvalue that corresponds to only one eigenvector. Even if δ is non-zero but small, this transformation also cannot be used, because the normalization of eigenvectors leads to large values of nonlinear coefficients, breaking the weak nonlinearity assumption. In this case, (4.10) must be performed with $\mathscr{Z}(k)$ in the form

$$\mathscr{Z}(k) = \begin{cases} \mathscr{Z}_e, & k > 0, \\ \mathscr{Z}_a, & k < 0, \end{cases}$$

where

$$\mathscr{Z}_e = \frac{\mathbf{z}_1 + \mathbf{z}_2}{2}, \quad \mathscr{Z}_a = \frac{\mathbf{z}_1 - \mathbf{z}_2}{\omega_1 - \omega_2}, \tag{4.16}$$

and $\boldsymbol{z}_1, \boldsymbol{z}_2$ are found in (4.13). It is easy to see that when the eigenvalues coalesce, this pair of vectors becomes the pair of eigenvector and generalized eigenvector. The normalization condition

$$(\mathscr{Z}_e^*, J\mathscr{Z}_a) = -\mathrm{i}$$

gives $c_1 = c_2 = \sqrt{A}$. Thus, in the instability region the transformation to generalized normal variables has the form (4.14 *a*, *b*), with

$$\phi_0(k) = -\frac{\mathrm{i}}{\sqrt{A}}\Theta(-k), \quad \eta_0(k) = \sqrt{A}\Theta(k). \tag{4.17}$$

It is easy to show that in these variables (2.24) has the canonical form

$$\dot{a}(k,t) = \mathrm{i}\ \mathrm{sign}\, k \frac{\delta H}{\delta a(-k,t)}, \tag{4.18}$$

and the density $\hat{H}_2$ of the leading-order (quadratic) term of the Hamiltonian is

$$\hat{H}_2 = -\frac{\omega_0}{2}\ \mathrm{sign}\ k a(k,t)a(-k,t) + \mathrm{c.c.} + \tilde{\Omega} a(k,t)a^*(k,t), \tag{4.19}$$

where $\omega_0 = (\omega_1 + \omega_2)/2$, and

$$\tilde{\Omega}(k) = \begin{cases} \delta^2, & k > 0, \\ 1, & k < 0; \end{cases}$$

note that in the instability region $\delta^2 < 0$.

In vector form,

$$\hat{H}_2 = \tfrac{1}{2}(\boldsymbol{a}(-k,t), \bar{\boldsymbol{h}}(k)\boldsymbol{a}(k,t)),$$

with $\boldsymbol{a}(k,t) = (a(k,t), a^*(-k,t))$, and $\bar{\boldsymbol{h}}(k)$ for $k > 0$ is

$$\bar{\boldsymbol{h}}(k) = \begin{pmatrix} -\omega_0 & 1 \\ \delta^2 & -\omega_0 \end{pmatrix}.$$

It is easy to see that if the eigenfrequencies ω_1 and ω_2 coalesce, i.e. $\delta^2 = 0$, then $\bar{\boldsymbol{h}}(k)$ has the form of the Jordan block.

This canonical structure is essentially different from the familiar structure for stable waves. In particular, the canonically conjugated variables are $a(k)$, $a(-k)$, and this indeed appears to be natural for the case of instability. It is worth noting that if the instability is not weak, we have the same canonical structure, but with a different form of the Hamiltonian H_2 (Goncharov & Pavlov 1993).

The canonical structure obtained is valid both for the stable and unstable cases, i.e. δ can be imaginary or real, but its absolute value should be small. So, the structure of dynamical equations and the form of quadratic Hamiltonian changes depending on the value of k.

Now we turn to the cubic term in the Hamiltonian expansion, responsible for the three-wave interaction. Substitution of (4.1), (4.2) into (2.23) and the subsequent Fourier transform gives the expression for H_3 in the form

$$H_3 = \frac{\mathrm{i}\Delta V}{2\sqrt{2\pi}} \int k_3(2|k_1| - |k_3|)\eta(k_1)\eta(k_2)\varphi(k_3)\delta(k_1 + k_2 + k_3)\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3. \tag{4.20}$$

In terms of $a(k)$, substituting (4.10) into (4.20),

$$H_3 = \frac{1}{\sqrt{2\pi}} \int [V(k_1,k_2,k_3)a(k_1)a(k_2)a^*(k_3)\delta(k_1 + k_2 - k_3) + \mathrm{c.c.}]\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3, \tag{4.21}$$

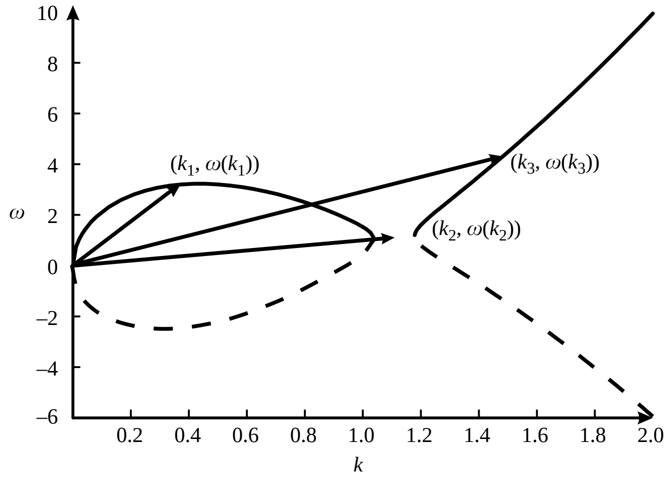

FIGURE 3. Dispersion relation (4.8), for $\sigma/\rho = 80\,\text{cm}^3\,\text{s}^{-2}$, $g\Delta\rho/\rho = 98\,\text{cm}\,\text{s}^{-2}$, $\Delta V = 1.001\Delta V_c$, $v = 1$. Dashed curves correspond to the negative sign of wave action density.

where non-resonant terms are omitted, and the interaction coefficient $V(k_1, k_2, k_3)$ is

$$
\begin{aligned}
V(k_1, k_2, k_3) = \frac{\mathrm{i}\Delta V}{2}[&k_3(2|k_1| - |k_3|)\eta_0(k_1)\eta_0(k_2)\phi_0^*(k_3) \\
&+ k_1(2|k_2| - |k_1|)\phi_0(k_1)\eta_0(k_2)\eta_0^*(k_3) \\
&+ k_2(2|k_3| - |k_2|)\eta_0(k_1)\phi_0(k_2)\eta_0^*(k_3)].
\end{aligned}
\tag{4.22}
$$

Here $\phi_0(k)$, $\eta_0(k)$ are defined by (4.15) or (4.17), again depending on the value of k.

4.3. *Resonant triplet for the KH model*

Having obtained the terms in the expansion of the Hamiltonian, we can now investigate the resonant interactions in the KH model.

As illustrated in figure 3, two types of synchronism conditions are fulfilled simultaneously,

$$k_{01} + k_{02} - k_{03} = 0, \quad \omega(k_{01}) + \omega(k_{02}) - \omega(k_{03}) = 0,$$

and

$$k_{01} - \hat{k}_{02} - k_{03} = 0, \quad \omega(k_{01}) - \omega(\hat{k}_{02}) - \omega(k_{03}) = 0, \quad \hat{k}_{02} = -k_{02}.$$

Note that in the KH model there are no explosive three-wave interactions.

4.4. *Evolution equations*

Let us consider the resonant interaction of three spectrally narrow wave packets centred at wavenumbers k_{01}, k_{02}, k_{03}, and write $a(k)$ in the form

$$a(k) = c_1(k_{01} + \kappa) + c_2(k_{02} + \kappa) + c_3(k_{03} + \kappa) + c_4(-k_{02} - \kappa). \tag{4.23}$$

We substitute (4.23) into (2.23) and ignore non-resonant terms that can be excluded by an appropriate canonical transformation. Then,

$$
\begin{aligned}
H_3 = \frac{2}{\sqrt{2\pi}} \int &[V(k_{01}, k_{02}, k_{03})c_1(k_{01} + \kappa_1)c_2(k_{02} + \kappa_2)c_3^*(k_{03} + \kappa_3) \\
&+ V(k_{01}, \hat{k}_{02}, k_{03})c_1^*(k_{01} + \kappa_3)c_4(-k_{02} - \kappa_2)c_3(k_{03} + \kappa_1) \\
&+ \text{c.c.}]\delta(\kappa_1 + \kappa_2 - \kappa_3)\,\mathrm{d}\kappa_1\,\mathrm{d}\kappa_2\,\mathrm{d}\kappa_3.
\end{aligned}
$$

Substituting (4.15), (4.17) into (4.22) and using the fact that k_{01}, k_{03} both lie in the stability region, while k_{02} is positive and lies in the instability region, we obtain the expression for $K = 2V(k_{01}, k_{02}, k_{03})$ in the form

$$K = \Delta V \sqrt{A(k_{02})} \left[k_{01}(2|k_{02}| - |k_{01}|) \left(\frac{C(k_{01})A(k_{03})}{A(k_{01})C(k_{03})} \right)^{1/4} + k_{03}(2|k_{01}| - |k_{03}|) \left(\frac{C(k_{03})A(k_{01})}{A(k_{03})C(k_{01})} \right)^{1/4} \right].$$

Now let us introduce the small parameter of nonlinearity ε, and assume that the linear instability effects and the effects of nonlinear interaction with neutrally stable waves are of the same order (i.e. $\varepsilon \sim \delta$). This leads to the scaling

$$c_1(k_{01}) \sim c_3(k_{03}) \sim \varepsilon^{3/2}, \quad c_2(k_{02}) \sim \varepsilon, \quad c_4(\hat{k}_{02}) \sim \varepsilon^2. \tag{4.24}$$

Then the second term in H_3 is of higher order in ε and can be neglected. However, the fourth-order term H_4 in the expansion of the Hamiltonian must be taken into account. The essential term of H_4 in terms of $a(k)$ has the form (Romanova 1998)

$$H_4 = \frac{\Gamma}{2\pi} \int c_2(k_{02} + \kappa_1) c_2(k_{02} + \kappa_2) c_2^*(k_{02} + \kappa_3) c_2^*(k_{02} + \kappa_4) \times \delta(\kappa_1 + \kappa_2 - \kappa_3 - \kappa_4)\, \mathrm{d}\kappa_1\, \mathrm{d}\kappa_2\, \mathrm{d}\kappa_3\, \mathrm{d}\kappa_4,$$

where $\Gamma = \Gamma(k_{02}, k_{02}, k_{02}, k_{02})$ is the coefficient of cubic nonlinear self-interaction. Now we can write the evolution equations for wave packets centred at the points k_{01}, k_{02}, $-k_{02}$, k_{03}, leaving only essential terms. For the points k_{01}, k_{03} we use the Hamiltonian equations in the form (3.22), while for the points k_{02}, $-k_{02}$ the form (4.18) is used. As a result we have

$$\dot{c}_1(k_{01} + \kappa) = -\mathrm{i}\omega(k_{01} + \kappa) c_1(k_{01} + \kappa) - \frac{\mathrm{i}}{\sqrt{2\pi}} K^* \int c_2^*(k_{02} + \kappa_2) c_3(k_{03} + \kappa_3) \delta(\kappa + \kappa_2 - \kappa_3)\, \mathrm{d}\kappa_2\, \mathrm{d}\kappa_3, \tag{4.25a}$$

$$\dot{c}_3(k_{03} + \kappa) = -\mathrm{i}\omega(k_{03} + \kappa) c_3(k_{03} + \kappa) - \frac{\mathrm{i}}{\sqrt{2\pi}} K \int c_1(k_{01} + \kappa_1) c_2(k_{02} + \kappa_2) \delta(\kappa_1 + \kappa_2 - \kappa)\, \mathrm{d}\kappa_1\, \mathrm{d}\kappa_2, \tag{4.25b}$$

$$\dot{c}_2(k_{02} + \kappa) = -\mathrm{i}\omega_0(k_{02} + \kappa) c_2(k_{02} + \kappa) + \mathrm{i} c_4^*(-k_{02} - \kappa), \tag{4.25c}$$

$$\begin{aligned} \dot{c}_4^*(-k_{02} - \kappa) = {} & -\mathrm{i}\omega_0(k_{02} + \kappa) c_4^*(-k_{02} - \kappa) + \mathrm{i}\delta^2 c_2(k_{02} + \kappa) \\ & + \frac{\mathrm{i}}{\sqrt{2\pi}} K^* \int c_1^*(k_{01} + \kappa_1) c_3(k_{03} + \kappa_3) \delta(\kappa_1 + \kappa - \kappa_3)\, \mathrm{d}\kappa_1\, \mathrm{d}\kappa_3 \\ & + \frac{2\mathrm{i}\Gamma}{2\pi} \int c_2(k_{02} + \kappa_1) c_2(k_{02} + \kappa_2) c_2^*(k_{02} + \kappa_3) \\ & \times \delta(\kappa_1 + \kappa_2 - \kappa_3 - \kappa)\, \mathrm{d}\kappa_1\, \mathrm{d}\kappa_2\, \mathrm{d}\kappa_3, \end{aligned} \tag{4.25d}$$

Let us introduce the 'slow' amplitudes of wave packets $Q_j(\kappa, T)$, where $T = \varepsilon t$. Using the scaling (4.24), we get

$$\left. \begin{aligned} c_1(k_{01} + \kappa, t) &= \varepsilon^{3/2} \exp(-\mathrm{i}\omega(k_{01})t) Q_1(\kappa, T), \\ c_3(k_{03} + \kappa, t) &= \varepsilon^{3/2} \exp(-\mathrm{i}\omega(k_{03})t) Q_3(\kappa, T), \\ c_2(k_{02} + \kappa, t) &= \varepsilon \exp(-\mathrm{i}\omega_0(k_{02})t) Q_2(\kappa, T), \\ c_4^*(-k_{02} - \kappa, t) &= \varepsilon^2 \exp(-\mathrm{i}\omega_0(k_{02})t) Q_4(\kappa, T). \end{aligned} \right\} \tag{4.26}$$

Then,

$$\frac{\partial Q_1(\kappa)}{\partial T} + \frac{\mathrm{i}\kappa}{\varepsilon} v_{1gr} Q_1(\kappa) = -\frac{\mathrm{i}}{\sqrt{2\pi}} K^* \int Q_2^*(\kappa_2) Q_3(\kappa_3) \delta(\kappa + \kappa_2 - \kappa_3)\, \mathrm{d}\kappa_2 \mathrm{d}\kappa_3, \quad (4.27a)$$

$$\frac{\partial Q_3(\kappa)}{\partial T} + \frac{\mathrm{i}\kappa}{\varepsilon} v_{3gr} Q_3(\kappa) = -\frac{\mathrm{i}}{\sqrt{2\pi}} K \int Q_1(\kappa_1) Q_2(\kappa_2) \delta(\kappa_1 + \kappa_2 - \kappa)\, \mathrm{d}\kappa_1 \mathrm{d}\kappa_2, \quad (4.27b)$$

$$\frac{\partial Q_2(\kappa)}{\partial T} + \frac{\mathrm{i}\kappa}{\varepsilon} v_{2gr} Q_2(\kappa) = \mathrm{i} Q_4(\kappa), \quad (4.27c)$$

$$\frac{\partial Q_4(\kappa)}{\partial T} + \frac{\mathrm{i}\kappa}{\varepsilon} v_{2gr} Q_4(\kappa) - \frac{\mathrm{i}\delta^2}{\varepsilon^2} Q_2(\kappa) = \frac{\mathrm{i}}{\sqrt{2\pi}} K^* \int Q_1^*(\kappa_1) Q_3^*(\kappa_3) \delta(\kappa_1 - \kappa - \kappa_3)\, \mathrm{d}\kappa_1\, \mathrm{d}\kappa_3$$
$$+ \frac{2\mathrm{i}\Gamma}{2\pi} \int Q_2(\kappa_1) Q_2(\kappa_2) Q_2^*(\kappa_3) \delta(\kappa_1 + \kappa_2 - \kappa_3 - \kappa)\, \mathrm{d}\kappa_1\, \mathrm{d}\kappa_2\, \mathrm{d}\kappa_3. \quad (4.27d)$$

Here

$$v_{1gr} = \frac{\partial \omega}{\partial k}(k_{01}), \quad v_{3gr} = \frac{\partial \omega}{\partial k}(k_{03}), \quad v_{2gr} = v = (V_1 + V_2)/2.$$

The inverse Fourier transform

$$\psi_j(X, T) = \frac{1}{\sqrt{2\pi}} \int Q_j(\kappa, T) \exp(\mathrm{i}\kappa x)\, \mathrm{d}\kappa$$

gives the evolution equations for ψ_j in the form

$$\left.\begin{aligned}
&\left(\frac{\partial}{\partial T} + v_{1gr}\frac{\partial}{\partial X}\right)\psi_1 + \mathrm{i}K^*\psi_3\psi_2^* = 0,\\
&\left(\frac{\partial}{\partial T} + v_{3gr}\frac{\partial}{\partial X}\right)\psi_3 + \mathrm{i}K\psi_1\psi_2 = 0,\\
&\left(\frac{\partial}{\partial T} + v_{2gr}\frac{\partial}{\partial X}\right)\psi_2 = \mathrm{i}\psi_4,\\
&\left(\frac{\partial}{\partial T} + v_{2gr}\frac{\partial}{\partial X}\right)\psi_4 = -\mathrm{i}V_0^2\left(\frac{\partial^2\psi_2}{\partial X^2} - \hat{b}k_0^2\psi_2\right) + \mathrm{i}K^*\psi_1^*\psi_3 + 2\mathrm{i}\Gamma|\psi_2|^2\psi_2,
\end{aligned}\right\} \quad (4.28)$$

where $X = \varepsilon x$, $\hat{b} = b/\varepsilon^2$. Excluding the term ψ_4 from the last two equations,

$$\left(\frac{\partial}{\partial T} + v_{2gr}\frac{\partial}{\partial X}\right)^2 \psi_2 - V_0^2\left(\frac{\partial^2\psi_2}{\partial X^2} - \hat{b}k_0^2\psi_2\right) + K^*\psi_1^*\psi_3 + 2\Gamma|\psi_2|^2\psi_2 = 0.$$

For the case of the interaction of three harmonic waves, when the amplitudes ψ_j do not depend on the spatial coordinate X, system (4.28) has the form

$$\dot{\psi}_1 + \mathrm{i}K^*\psi_2^*\psi_3 = 0,$$

$$\dot{\psi}_3 + \mathrm{i}K\psi_1\psi_2 = 0,$$

$$\dot{\psi}_2 - \mathrm{i}\psi_4 = 0,$$

$$\dot{\psi}_4 = \mathrm{i}\hat{\delta}^2\psi_2 + \mathrm{i}K^*\psi_1^*\psi_3 + 2\mathrm{i}\Gamma|\psi_2|^2\psi_2,$$

where $\hat{\delta}^2 = \hat{b}k_0^2V_0^2$. Again, negative values of $\hat{b}$ correspond to the instability case.

After the substitution $\psi_j = B_j/K$,

$$\left.\begin{aligned} &\dot{B}_1 + \mathrm{i}B_2^* B_3 = 0, \\ &\dot{B}_3 + \mathrm{i}B_1 B_2 = 0, \\ &\dot{B}_2 - \mathrm{i}B_4 = 0, \\ &\dot{B}_4 = \mathrm{i}\hat{\delta}^2 B_2 + \mathrm{i}B_1^* B_3 + 2\mathrm{i}R|B_2|^2 B_2, \end{aligned}\right\} \tag{4.29}$$

where $R = \Gamma/|K|^2$. This system is easily shown to be Hamiltonian,

$$\left.\begin{aligned} \dot{B}_1 &= -\mathrm{i}\frac{\partial H}{\partial B_1^*}, \quad & \dot{B}_3 &= -\mathrm{i}\frac{\partial H}{\partial B_3^*}, \\ \dot{B}_2 &= \mathrm{i}\frac{\partial H}{\partial B_4^*}, \quad & \dot{B}_4 &= \mathrm{i}\frac{\partial H}{\partial B_2^*}, \end{aligned}\right\} \tag{4.30}$$

with the Hamiltonian of the form

$$I_1 = |B_4|^2 + \hat{\delta}^2|B_2|^2 + B_1^* B_2^* B_3 + B_1 B_2 B_3^* + R|B_2|^4. \tag{4.31}$$

The other two conservation laws are

$$I_2 = |B_1|^2 + |B_3|^2, \tag{4.32}$$

and

$$I_3 = B_2 B_4^* + B_2^* B_4 + |B_1|^2. \tag{4.33}$$

4.5. *Discussion*

In this and in the preceding section, we have considered two different models of resonant interactions involving weakly unstable waves, and obtained two different sets of evolution equations.

At first glance, the two models may seem equivalent at a linear level. In particular, if the nonlinear terms in (3.35) and (4.29) are omitted, then there is no wave interaction, and both systems reduce to the same second-order equation. However, this does not mean that they are equivalent. In order to demonstate this, let us first consider the system (3.32). Linearization gives

$$\left.\begin{aligned} \dot{c}_2 &= -\mathrm{i}\omega c_2 - \mathrm{i}f_1 c_4^*, \\ \dot{c}_4^* &= -\mathrm{i}\omega c_4^* + \mathrm{i}f_1 c_2. \end{aligned}\right\} \tag{4.34}$$

Here, we have put $\kappa = 0$ and denoted $\omega = -\omega_1(-k_{02}) = \omega_2(k_{02})$, $c_2 = c_2(k_{02})$, $c_4^* = c_4^*(-k_{02})$. Then, in vector form, with $c = (c_2, c_4^*)$,

$$\dot{\boldsymbol{c}} = -\mathrm{i}\boldsymbol{A}\boldsymbol{c}, \tag{4.35}$$

where the matrix $\boldsymbol{A}$ is

$$\boldsymbol{A} = \begin{pmatrix} \omega & f_1 \\ -f_1 & \omega \end{pmatrix}. \tag{4.36}$$

If $f_1 = 0$, double eigenfrequency ω corresponds to two linearly independent eigenvectors, which are obtained from the equation $(\boldsymbol{A} - \omega\boldsymbol{E})c = 0$, $\boldsymbol{E}$ being an identity matrix. If $f_1 \neq 0$, the solution exhibits exponential growth.

Let us now turn to the second model. Omitting nonlinear terms in (4.26), we get

$$\left.\begin{aligned}\dot{c}_2 &= -\mathrm{i}\omega c_2 + \mathrm{i}c_4^*, \\ \dot{c}_4^* &= -\mathrm{i}\omega c_4^* + \mathrm{i}\delta^2 c_2.\end{aligned}\right\} \tag{4.37}$$

Again, we have put $\kappa = 0$ and denoted $\omega = \omega_0(k_{02})$, $c_2 = c_2(k_{02})$, $c_4^* = c_4^*(-k_{02})$. Then, in vector form

$$\dot{c} = -\mathrm{i}\hat{\boldsymbol{A}}c, \tag{4.38}$$

where $c = (c_2, c_4^*)$, and

$$\hat{\boldsymbol{A}} = \begin{pmatrix} \omega & -1 \\ -\delta^2 & \omega \end{pmatrix}. \tag{4.39}$$

When the parameter of supercriticality δ is equal to zero, two eigenvalues coincide, $\hat{\boldsymbol{A}}$ has the structure of the Jordan block, and has only one eigenvector. Then, the solution has the form (Gelfand 1989)

$$c = \hat{\boldsymbol{c}}_1 t \exp(-\mathrm{i}\omega t) + \hat{\boldsymbol{c}}_2 \exp(-\mathrm{i}\omega t), \tag{4.40}$$

where the multiple eigenfrequency is equal to ω, and $\hat{\boldsymbol{c}}_1$ and $\hat{\boldsymbol{c}}_2$ are the eigenvector and the generalized eigenvector of $\hat{\boldsymbol{A}}$ respectively, satisfying the equations

$$(\hat{\boldsymbol{A}} - \omega E)\hat{\boldsymbol{c}}_1 = 0, \quad (\hat{\boldsymbol{A}} - \omega E)\hat{\boldsymbol{c}}_2 = \mathrm{i}\hat{\boldsymbol{c}}_1. \tag{4.41}$$

The solution remains unstable when δ^2 is equal to zero, but now this instability is algebraic.

To summarize, the mathematical difference between the two models is due to the invariance of the normal form of the matrix, which appears in the second model (i.e. it cannot be reduced to diagonal form by any linear transformation). Physically, this difference is manifested by the fact that the second model remains unstable (algebraically) even at zero value of the supercriticality parameter. Another important physical consequence is that the scaling of amplitudes in the two models is different (cf. (3.33) and (4.26)). In order to obtain the amplitude equations, it is necessary to assume that the effects of weak nonlinearity and weak instability are of the same order. In the second model, this assumption leads to the scaling (4.24), while in the first model, all the amplitudes have the same order. It is also important to compare the two Hamiltonian structures. In the first model, the Hamiltonian structure is the same as in the case of stable waves, while in the second model it is fundamentally different. Finally, the presence of algebraic instability in the second model leads to much richer behaviour of solutions. The complete numerical analysis would require a special study beyond the scope of the present paper. In the next section, we will consider an example for the most interesting case of zero supercriticality.

5. Evolution of a resonant triplet in the KH model

Pedlosky (1975) and Weissman (1978) showed that the nonlinear cubic term in the Klein–Gordon equation that describes the dynamics of a weakly unstable wave packet in the KH model is destabilizing, provided that the density difference is large enough. In terms of the density ratio, this condition can be written as $\rho_1/\rho_2 < 0.283$. In the present work, we assume that the Boussinesq approximation is applicable, therefore, $\rho_1/\rho_2 \sim 1$, and the cubic nonlinearity has a stabilizing effect. In order to find out whether the quadratic interaction with neutral modes has a stabilizing or destabilizing effect, we neglect the cubic nonlinearity, assuming $R = 0$. In addition, we assume that the parameter of supercriticality $\hat{\delta}$ is zero, so that the instability is algebraic. According to the scaling (4.24), the effects of the algebraic instability appear at the same order as the effects of the nonlinear interaction.

Let us rewrite (4.29) in terms of real variables r_j, ϕ_j,

$$B_j = r_j \exp(\mathrm{i}\phi_j),$$

taking into account the conservation law (4.32) and introducing the new dependent variable λ according to relations $r_1 = a \sin\lambda$, $r_3 = a\cos\lambda$, where $a^2 = I_2$, and two phase variables $\theta_1 = \phi_3 - \phi_1 - \phi_2$ and $\theta_2 = \phi_2 - \phi_4$.

In this way, we obtain the system of equations

$$\dot{\lambda} = r_2 \sin\theta_1, \tag{5.1a}$$

$$\dot{r}_2 = r_4 \sin\theta_2, \tag{5.1b}$$

$$\dot{r}_4 = -\frac{a^2}{2}\sin(2\lambda)\sin(\theta_1+\theta_2), \tag{5.1c}$$

$$\dot{\theta}_1 = 2r_2\frac{\cos(2\lambda)}{\sin(2\lambda)}\cos\theta_1 - \frac{r_4}{r_2}\cos\theta_2 = 0, \tag{5.1d}$$

$$\dot{\theta}_2 = \frac{r_4}{r_2}\cos\theta_2 - \frac{a^2}{2r_4}\sin(2\lambda)\cos(\theta_1+\theta_2). \tag{5.1e}$$

In terms of new variables, conservation laws (4.31), (4.33) have the form

$$\left.\begin{aligned} r_4^2 + a^2\sin(2\lambda)\cos\theta_1 r_2 = I_1,\\ 2r_2r_4\cos\theta_2 + a^2\sin^2\lambda = I_3.\end{aligned}\right\} \tag{5.2}$$

Consider the case when the amplitude of neutral waves is small, i.e. $a \ll 1$. Then, system (5.1) can be studied analytically, due to the wide separation of timescales. It is easy to see that while λ and θ_1 have a fast time dependence, the variables r_2, r_4 and θ_2 depend on the slow timescale.

It is useful to denote $Y = \sin(2\lambda)\cos\theta_1$, $Z = \sin(2\lambda)\sin\theta_1$. Then, it follows from (5.1) that

$$\dot{Y} = -\frac{r_4}{r_2}\cos\theta_2 Z \tag{5.3}$$

and

$$\dot{Z} = 2r_2\cos(2\lambda) - \frac{r_4}{r_2}\cos\theta_2 Y. \tag{5.4}$$

Now, let us first consider the case $r_2 \gg r_4$. Then, $r_4\cos\theta_2/r_2$ is small and can be neglected, leading to approximate conservation of Y,

$$Y \simeq I_0,$$

and

$$\dot{Z} = 2r_2\cos(2\lambda). \tag{5.5}$$

On the other hand, from (5.1),

$$\frac{\mathrm{d}}{\mathrm{d}t}(r_4\sin\theta_2) = -\frac{a^2}{2}Y, \quad \frac{\mathrm{d}}{\mathrm{d}t}(r_4\cos\theta_2) = -\frac{a^2}{2}Z. \tag{5.6}$$

Equations (5.6) and the conservation of Y give

$$r_4\sin\theta_2 = -\frac{a^2}{2}I_0(t-t_0), \tag{5.7}$$

and

$$r_2 = -\frac{a^2}{2} I_0 (t - t_0)^2 + r_{2\max}, \tag{5.8}$$

i.e. the form of r_2 is parabolic in the interval between two minimal values of r_2. At the critical point, where $r_2 = r_{2\max}$, $\dot{r}_{2,4} = 0$, $\dot{\lambda} = 0$, $\theta_{1,2} = 0$. Let us consider a small vicinity of this point. The solution of (5.5) is

$$Z = \sqrt{1 - I_0^2} \sin\left[2 r_{2\max}(t - t_0) - \frac{a^2}{6} I_0 (t - t_0)^3\right], \tag{5.9}$$

and, using (5.6),

$$r_4 \cos\theta_2 \approx \frac{a^2\sqrt{1 - I_0^2}}{4 r_{2\max}} [\cos(2 r_{2\max}(t - t_0)) - 1] + r_{4\min} \tag{5.10}$$

and it follows from (5.7), (5.10) that

$$r_4 \approx r_{4\min}\left(1 + \frac{a^4 I_0^2 (t - t_0)^2}{4 r_{4\min}^2} + \frac{a^2\sqrt{1 - I_0^2}(\cos(2 r_{2\max}(t - t_0)) - 1)}{2 r_{2\max} r_{4\min}}\right)^{1/2} \tag{5.11}$$

and

$$\theta_2 \approx -\arctan\left(\frac{a^2 I_0 (t - t_0)}{2 r_{4\min}}\left[1 - \frac{a^2\sqrt{1 - I_0^2}(\cos(2 r_{2\max}(t - t_0)) - 1)}{2 r_{2\max} r_{4\min}}\right]\right). \tag{5.12}$$

System (5.1) was solved numerically for $a = 0.1\sqrt{2}$ and the initial conditions $r_2 = r_4 = 1$, $\lambda = \pi/2$, $\theta_1 = \theta_2 = 0$ at $t = 0$. Figure 4 shows the numerical solution for 'slow' variables r_2, r_4 and θ_2. The evolution of Y is shown in figure 5. We see that the condition $r_2 \gg r_4$ is fulfilled almost everywhere, except for the vicinity of transfer points. Between these points, Y is conserved, playing the role of the 'intermediate' integral of motion, and the variable r_2 has the parabolic form, with the maximum being defined by the value of Y. At this point of maximum of r_2, r_4 has minimum, and the 'slow' phase θ_2 changes sign from $-\pi/2$ to $\pi/2$. The detailed view of this behaviour in the vicinity of the point $t_0 = 116$ is presented in figure 6. According to the solution (5.11), r_4 has small fast oscillations at the minimum point. The fast variables λ and θ_1 depend on a much smaller timescale.

At the other critical point, where r_2 has a minimum and r_4 has a maximum, r_2 and r_4 are comparable, the expression $r_4 \cos\theta_2 / r_2$ is not small and Y is not constant. Then, since $a \ll 1$, we neglect the second term in (5.1 e), obtaining the pair of equations

$$\dot{r}_2 = r_4 \sin\theta_2, \quad r_2\dot{\theta}_2 = r_4 \cos\theta_2. \tag{5.13}$$

Assuming that r_4 changes slowly, $r_4 \approx r_{4\max}$, the solution of (5.13) is

$$\theta_2 = \arctan\left(\frac{r_{4\max}}{r_{2\min}}(t - t_1)\right), \quad r_2 = r_{2\min}(1 + r_{4\max}(t - t_1)^2)^{1/2}.$$

These analytical expressions agree well with the numerical solution of the system (5.1) in the vicinity of the point $t_1 = 232$ (figure 7). Note that at this point, the wide separation of timescales ceases to be valid, and the timescales for the 'slow' and 'fast' variables become comparable. The neighbourhood of t_1 is the region where Y changes rapidly, in a transition from one stability level to another. We can consider the level of stabilization for Y as a random value lying in the interval between 0 and 1. Thus, the amplitudes of the oscillations of r_2 shown in figure 4 for a large time interval are

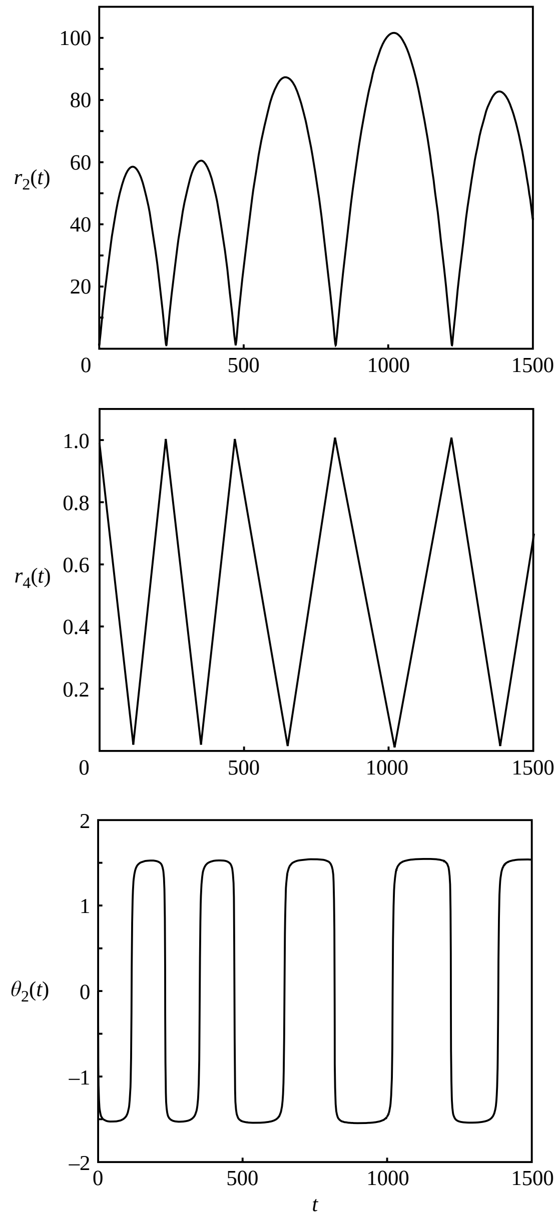

FIGURE 4. Time evolution of 'slow' variables r_2, r_4 and θ_2, obtained by the numerical solution of (5.1) for $a = 0.1\sqrt{2}$ and initial conditions $r_2 = r_4 = 1$, $\lambda = \pi/2$, $\theta_1 = \theta_2 = 0$.

stochastic, their value depending on the value at which the function $Y(t)$ is stabilized after the narrow region where it oscillates quickly.

6. Concluding remarks

We have considered the dynamics of unstable disturbances in resonant interaction with stable ones. Throughout the paper, we assumed, without the loss of generality, that the linear instability is weak (otherwise, the account for nonlinear processes does

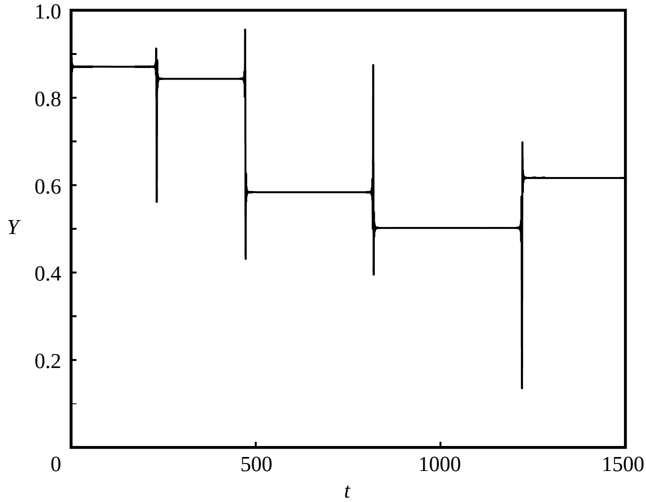

FIGURE 5. Evolution of $Y(t) = \sin(2\lambda)\cos\theta_1$. Initial conditions are as in figure 4.

not make sense), and that the dispersion relation admits only one instability region (this is the generic situation). We have demonstrated that the diversity of three-wave interactions involving one weakly unstable wave packet is reduced to two fundamental models.

The first one is exemplified by the three-layer model of stratified shear flow when interfaces are sufficiently separated in order to ensure the weak coupling of wave perturbations. The second model is represented by the Kelvin–Helmholtz model when the velocity shear is close to its critical value. With the method of Hamiltonian formalism, all the features specific to the model considered are contained in the expressions for the coefficients of interaction, so that the evolution equations for the amplitudes of resonantly interacting wave packets are universal and valid for waves of any type.

From the hydrodynamical viewpoint, the fundamental question is the possibility of the further growth of linearly unstable waves. Can the amplitude of a linearly unstable wave be stabilized due to nonlinear interactions with neutral waves? In the context of the first model, the positive answer was found earlier, for a particular problem, by Loesch (1974) and Pedlosky (1975) for baroclinic Rossby waves on the β-plane. For the second model, the answer was unknown. Analytical and numerical solutions of the system of equations derived for the evolution of amplitudes demonstrated the nonlinear stabilization of the linearly unstable wave. Moreover, the formalism developed in this paper allows a much wider class of models to be considered. For instance, a wave system consisting of a large number of resonant triads can be considered, and it is reasonable to suppose that the presence of triads consisting of stable waves will lead to further stabilization.

The results of this paper can be used for the interpretation of numerical simulations obtained with numerical models of the ocean and atmosphere, which are usually based on multilayer vertical structure. On the other hand, the results can be easily generalized to continuous stratification, provided that the basic scaling is conserved. It is also possible to generalize the results to three-dimensional flows. This task is non-trivial and is the subject of further study.

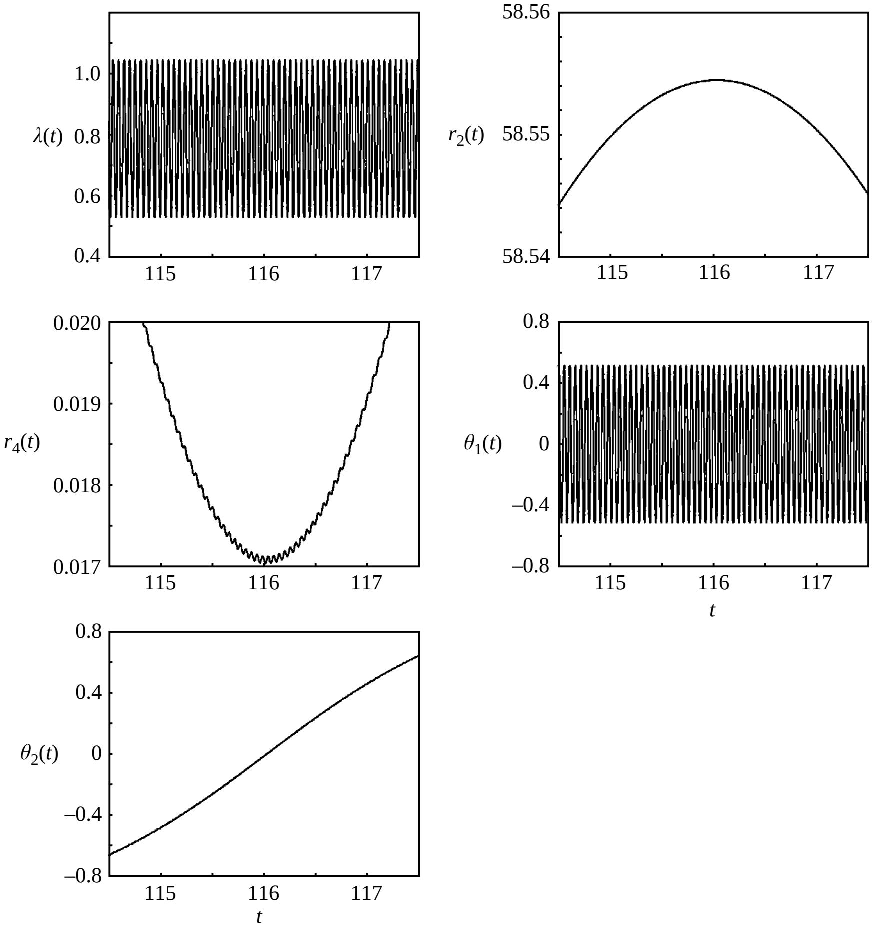

FIGURE 6. Time evolution of 'slow' variables $r_2(t)$, $r_4(t)$, $\theta_2(t)$ and 'fast' variables $\lambda(t)$, $\theta_1(t)$ in (5.1) in the vicinity of the point $t_0 = 116$ (maximum of $r_2(t)$), for the same initial conditions as in figure 4.

This paper was confined to the study of the dynamics of weakly unstable disturbances. However, it is straightforward to apply the results to the kinetics of linearly unstable wave fields, and thus to describe the development of instabilities in the presence of broadband noise. For stable wave systems, a statistical description of wave field is provided by the well-known formalism based on the hypothesis of quasi-Gaussianity and leading to the so-called kinetic or Boltzmann equation for second statistical moments of the field (Zakharov, L'vov & Falkovich 1992). In an unstable medium, this classical formalism is not applicable. However, the statistical description of weakly unstable wave fields can be achieved by direct numerical simulation, using the interaction coefficients derived in this paper.

This work was supported in part by RFBR grants No 04-05-64877 and No 04-05-64784, and by INTAS grant 01-234. We are grateful to A. Levin and I. Yakushkin for numerous stimulating discussions, to V. I. Shrira for valuable comments on the first draft of the manuscript, and to anonymous referees whose suggestions were very helpful.

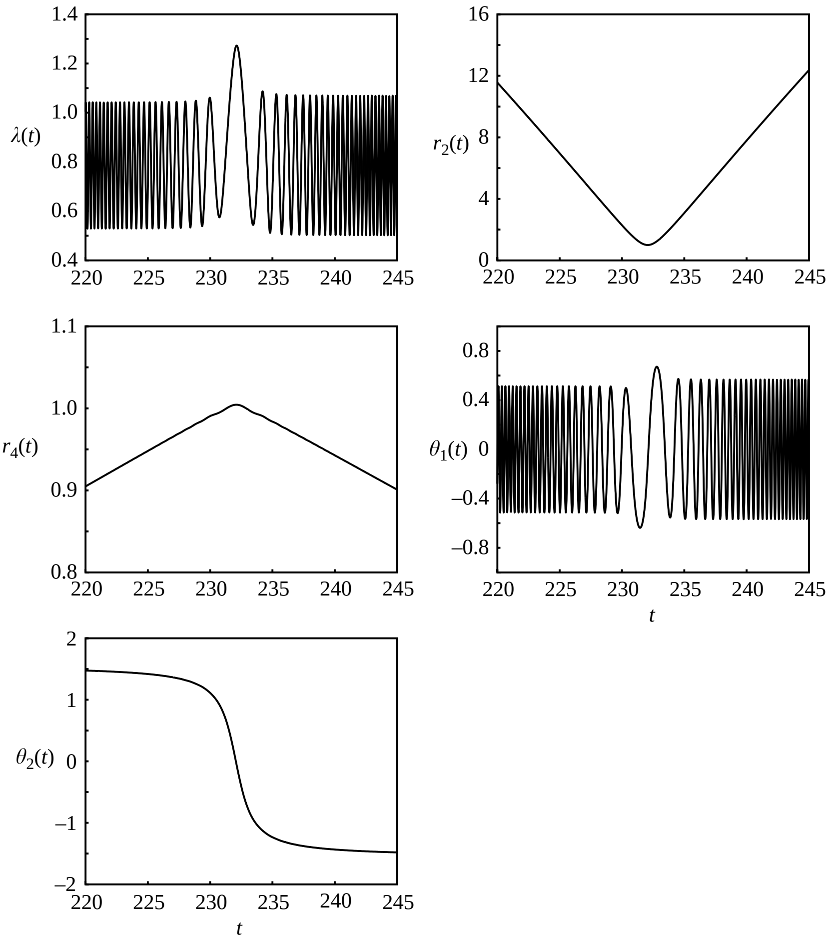

FIGURE 7. Time evolution of 'slow' variables $r_2(t)$, $r_4(t)$, $\theta_2(t)$ and 'fast' variables $\lambda(t)$, $\theta_1(t)$ in (5.1) in the vicinity of the point $t_1 = 232$ (minimum of r_2), for the same initial conditions as in figure 4.

Appendix. Interaction coefficients in the three-layer model

In terms of a Fourier transform, the cubic term H_3 in the expansion of the Hamiltonian has the form (3.7). It can be shown that H_3 can be written as

$$H_3 = \frac{1}{\sqrt{2\pi}} \int \sum_{j=1}^{2} \Big\{ \big(\tfrac{1}{2}\nu_j \eta_j(k_1) - \mathrm{i}k_1 \phi_j(k_1)\big)\eta_j(k_2)\big(p_j(k_3) - \tfrac{1}{2}V'_{0j}\eta_j(k_3)\big) - \tfrac{1}{12}V'_{0j}\nu_j \eta_j(k_1)\eta_j(k_2)\eta_j(k_3)\Big\} \delta(k_1+k_2+k_3)\,\mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3. \qquad \text{(A 1)}$$

Let us introduce the notation (omitting indices)

$$Q(k) = p(k) - \tfrac{1}{2}V'_0\eta(k), \quad R(k) = \tfrac{1}{2}\nu\eta(k) - \mathrm{i}k\phi(k). \qquad \text{(A 2)}$$

Then, (A 1) takes the form

$$H_3 = \frac{1}{\sqrt{2\pi}} \int \sum_{j=1}^{2} G_j(k_1, k_2, k_3)\delta(k_1+k_2+k_3)\,\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3, \qquad \text{(A 3)}$$

where

$$G_j(k_1, k_2, k_3) = R_j(k_1)\eta_j(k_2)Q_j(k_3) - \tfrac{1}{12}\nu_j V'_{0j}\eta_j(k_1)\eta_j(k_2)\eta_j(k_3). \qquad \text{(A 4)}$$

In terms of normal variables $a_j(k)$, the multipliers in the expression for G_j are

$$R_j(k, t) = \sum_{i=1}^{2}(r_j(k, \omega_i(k))a_i(k, t) + r_j^*(-k, \omega_i(-k))a_i^*(-k, t)), \qquad \text{(A 5}a)$$

$$\eta_j(k, t) = \sum_{i=1}^{2}(\eta_{0j}(k, \omega_i(k))a_i(k, t) + \eta_{0j}^*(-k, \omega_i(-k))a_i^*(-k, t)), \qquad \text{(A 5}b)$$

$$Q_j(k, t) = \sum_{i=1}^{2}(q_j(k, \omega_i(k))a_i(k, t) + q_j^*(-k, \omega_i(-k))a_i^*(-k, t)). \qquad \text{(A 5}c)$$

Coefficients r_j, q_j, η_{0j} are obtained by the substitution of the expressions for components of the eigenvector (3.12) into (A 2),

$$q_j(k) = n_j(k)\eta_{0j}(k), \qquad \text{(A 6)}$$

where

$$n_1(k) = -\tfrac{1}{2}(k\varepsilon(k)b_1(k)\eta_{02}(k)/\eta_{01}(k) + V'_{01}),$$
$$n_2(k) = \tfrac{1}{2}(k\varepsilon(k)b_1(k)\eta_{01}(k)/\eta_{02}(k) - V'_{02}).$$

Coefficient $r_j(k)$ is

$$r_j(k) = m_j(k)\eta_{0j}(k), \qquad \text{(A 7)}$$

where $m_j = \nu_j/2 - kb_j$.

In what follows, it is important that the values of $n_j(k)$ and $m_j(k)$ are always real and have the order of unity, while the values of $\eta_{0j}(k)$ are purely imaginary. Let us write the integrand in H_3 in terms of normal variables. We substitute (A 5*a–c*) into (A 4), leaving then only the terms proportional to $a_1(k_1)a_1(k_2)a_2(k_3)$, $a_1(k_1)a_2^*(k_2)a_2(k_3)$ and their complex conjugates, since these terms cannot be excluded by a canonical transformation (see Zakharov 1974; Kuznetsov & Zakharov 1997). As a result, we obtain for H_3

$$H_3 = \frac{1}{\sqrt{2\pi}}\int\left\{\sum_{j=1}^{2}\mathscr{A}_j(k_1, k_2, k_3)a_1(k_1)a_1(k_2)a_2(k_3)\delta(k_1 + k_2 + k_3)\right.$$
$$\left. + \sum_{j=1}^{2}\mathscr{B}_j(k_1, k_2, k_3)a_1(k_1)a_2^*(k_2)a_2(k_3)\delta(k_1 - k_2 + k_3) + \text{c.c.}\right\} \mathrm{d}k_1\,\mathrm{d}k_2\,\mathrm{d}k_3. \qquad \text{(A 8)}$$

Using (A 6), (A 7), we obtain for $\mathscr{A}_j$ and $\mathscr{B}_j$

$$\mathscr{A}_j = \eta_{0j}(k_1, \omega_1)\eta_{0j}(k_2, \omega_1)\eta_{0j}(k_3, \omega_2)[m_j(k_1, \omega_1)n_j(k_2, \omega_1)$$
$$+ m_j(k_2, \omega_1)n_j(k_3, \omega_2) + m_j(k_1, \omega_1)n_j(k_2, \omega_1) - \nu_j V'_{0j}/4]$$

and

$$\mathscr{B}_j = \eta_{0j}(k_1, \omega_1)\eta_{0j}^*(k_2, \omega_2)\eta_{0j}(k_3, \omega_2)[m_j(k_1, \omega_1)n_j^*(k_2, \omega_2)$$
$$+ m_j(k_1, \omega_1)n_j(k_3, \omega_2) + m_j^*(k_2, \omega_2)n_j(k_1, \omega_1) + m_j^*(k_2, \omega_2)n_j(k_3, \omega_2)$$
$$+ m_j(k_3, \omega_2)n_j(k_1, \omega_1) + m_j(k_3, \omega_2)n_j^*(k_2, \omega_2) - \nu_j V'_{0j}/4].$$

In order to obtain the evolution equations for the amplitudes of the resonantly interacting wave packets, the interaction coefficients should be calculated at the midpoints (k_{01}, k_{02}, k_{03}) and $(k_{01}, \hat{k}_{02}, k_{03})$.

Provided that the coupling is weak, the points k_{02}, k_{03} lie in the region of large wavenumbers, so we have

$$\eta_{02}(\hat{k}_{02}, \omega_1) = 0, \quad \eta_{02}(k_{02}, \omega_1) = 0, \quad \eta_{01}(k_{03}, \omega_2) \sim \varepsilon(k_{03}). \tag{A 9}$$

Note that the wavenumber k_{01} lies in the region of small k (i.e. long waves), where the coupling of modes is strong, so that $\eta_{02}(k_1, \omega_1) = \mathrm{i}\sqrt{D_2}/(b_2\sqrt{L})$ is not small. It follows from (A 9) that the coefficient $\mathscr{A}_2(k_{01}, \hat{k}_{02}, k_{03})$ is equal to zero, and the coefficient $\mathscr{A}_1(k_{01}, \hat{k}_{02}, k_{03})$, corresponding to explosive interaction, has the order of $\varepsilon(k_{03})$ and can be neglected. This means that the effects of explosive instability in the model considered are negligibly small. Coefficient $\mathscr{B}_1(k_{01}, k_{02}, k_{03})$ is also equal to zero, but the coefficient $\mathscr{B}_2(k_{01}, k_{02}, k_{03})$ has the order of unity. Thus, this is the only coefficient to be retained in (A 8), and we obtain the expression for H_3 in the form

$$H_3 = \frac{1}{\sqrt{2\pi}} \int [\mathscr{B}_2(k_1, k_2, k_3) a_1(k_1) a_2^*(k_2) a_2(k_3) \delta(k_1 - k_2 + k_3) + \text{c.c.}]\, \mathrm{d}k_1\, \mathrm{d}k_2\, \mathrm{d}k_3. \tag{A 10}$$

Considering the coefficient at the point (k_{01}, k_{02}, k_{03}) and using the conditions (A 9), we obtain

$$n_2(k_{02}, \omega_2) = -V'_{02}/2, \quad n_2(k_{03}, \omega_2) \sim -V'_{02}/2,$$

and

$$n_2(k_{01}, \omega_2) = -V'_{02}/2 + k_{01}\varepsilon(k_{01}) b_1(k_{01}, \omega_1)\eta_{01}(k_{01}, \omega_1)/\eta_{02}(k_{01}, \omega_1).$$

As a result, we get the following expression for $\mathscr{B}_2(k_{01}, k_{02}, k_{03})$:

$$\begin{aligned}\mathscr{B}_2(k_{01}, k_{02}, k_{03}) = {} & T_1 \eta_{02}(k_{01}, \omega_1)\eta_{02}^*(k_{02}, \omega_2)\eta_{02}(k_{03}, \omega_2) \\ & - T_2 \eta_{01}(k_{01}, \omega_1)\eta_{02}^*(k_{02}, \omega_2)\eta_{02}(k_{03}, \omega_2),\end{aligned}$$

where

$$T_1 = V'_{02}\left[\nu_2 + g\lambda_2\left(\frac{k_{01}}{\omega_1(k_{01}) - V_2 k_{01}} + \frac{k_{02}}{\omega_2(k_{02}) - V_2 k_{02}} + \frac{k_{03}}{\omega_2(k_{03}) - V_2 k_{03}}\right)\right]$$

and

$$\begin{aligned}T_2 = {} & \frac{\varepsilon(k_{01})}{2}\left(\nu_1 + g\lambda_1 \frac{k_{01}}{\omega_1(k_{01}) - V_1 k_{01}}\right) \\ & \times \left[\nu_2 + g\lambda_2\left(\frac{k_{02}}{\omega_2(k_{02}) - V_2 k_{02}} + \frac{k_{03}}{\omega_2(k_{03}) - V_2 k_{03}}\right)\right].\end{aligned}$$

Expressions for $\eta_{01}(k_{01}, \omega_1)$, $\eta_{02}(k_{01}, \omega_1)$, $\eta_{02}(k_{02}, \omega_2)$, $\eta_{02}(k_{03}, \omega_2)$ can be written as

$$\eta_{01}(k_{01}, \omega_1) = \frac{\mathrm{i}\sqrt{D_2(k_{01})}}{b_1(k_{01})\sqrt{L(k_{01})}} = \frac{\mathrm{i}}{b_1(k_{01})\sqrt{D'_1(k_{01}) + D'_2(k_{01})D_2(k_{01})/D_1(k_{01})}},$$

$$\eta_{02}(k_{01}, \omega_1) = -\frac{\mathrm{i}\sqrt{D_1(k_{01})}}{b_2(k_{01})\sqrt{L(k_{01})}} = -\frac{\mathrm{i}}{b_2(k_{01})\sqrt{D'_2(k_{01}) + D'_1(k_{01})D_1(k_{01})/D_2(k_{01})}},$$

$$\eta_{02}(k_{02}, \omega_2) = -\frac{\mathrm{i}}{b_2(k_{02})\sqrt{D'_2(k_{02})}}, \qquad \eta_{02}(k_{03}, \omega_2) = -\frac{\mathrm{i}}{b_2(k_{03})\sqrt{D'_2(k_{03})}}.$$

REFERENCES

Benjamin, T. B. & Bridges, T. J. 1997 Reappraisal of the Kelvin-Helmholtz problem. Part 1. Hamiltonian structure. *J. Fluid Mech.* **333**, 301–325.

Cairns, R. A. 1979 The role of negative energy waves in some instabilities of parallel flow. *J. Fluid Mech.* **92**, 1–14.

Craik, A. D. D. 1985 *Wave Interactions and Fluid Flows*. Cambridge University Press.

Craik, A. D. D. & Adam, J. A. 1979 Explosive resonant wave interaction in a three-layer fluid flow. *J. Fluid Mech.* **92**, 15–33.

Drazin, P. G. 1970 Kelvin-Helmholtz instability of finite amplitude. *J. Fluid Mech.* **17**, 321–335.

Gelfand, I. M. 1989 *Lectures on Linear Algebra*. Dover.

Goncharov, V. P. 1986 Nonlinear waves in flows that are vorticity-homogeneous in a layerwise manner. *Sov. Phys. Izv. Atm. Ocean Phys.* **22**, 355–361.

Goncharov, V. P. & Pavlov, V. I. 1993 *Problems of Hydrodynamics in Hamiltonian Description*. Moscow State University (in Russian).

Grimshaw, R. H. G. 1994 Resonant wave interaction near a critical level in a stratified shear flow. *J. Fluid Mech.* **269**, 1–22.

Grimshaw, R. & Christodoulides, P. 2001 Short-wave instability in a three-later stratified shear flow. *Q. J. Mech. Appl. Maths* **54**, 375–388.

Kaup, D. J., Reiman, A. & Bers, A. 1979 Space-time evolution of nonlinear three-wave interactions. I. Interactions in a homogeneous medium. *Rev. Mod. Phys.* **51**, 275–310.

Kuznetsov, E. A. & Zakharov, V. E. 1997 Hamiltonian formalism for nonlinear waves. *Physics–Uspekhi* **40**, 1087–1116.

Loesch, A. Z. 1974 Resonant interactions between unstable and neutral baroclinic waves. Parts I–II. *J. Atmos. Sci.* **31**, 1137–1217.

Pedlosky, J. 1975 The amplitude of baroclinic wave triades and mesoscale motion in the ocean. *J. Phys. Oceanogr.* **5**, 608–614.

Ripa, P. 1981 Symmetries and conservation laws for internal gravity waves. In *Nonlinear Properties of Internal Waves* (ed. B. J. West). AIP Conf. Proc. vol. 76, pp. 281–306.

Romanova, N. N. 1994 Hamiltonian description of wave dynamics in nonequilibrium media. *Nonlin. Proc. Geophys.* **1**, 234–240.

Romanova, N. N. 1998 Hamiltonian approach to the derivation of evolution equations for wave trains in weakly unstable media. *Nonlin. Proc. Geophys.* **5**, 241–253.

Romanova, N. N. & Yakushkin, I. G. 2001 Hamiltonian description of motion in a perfect stratified fluid. *Dokl. Phys.* **46**, 742–746.

Schmid, P. J. & Henningson, D. S. 2001 *Stability and Transition in Shear Flows*. Springer.

Triantafyllou, G. S. 1994 Note on the Kelvin–Helmholtz instability of stratified fluids. *Phys. Fluids* **6**, 164–171.

Vanneste, J. 1996 Rossby wave interaction in a shear flow with critical levels. *J. Fluid Mech.* **323**, 317–338.

Virasoro, M. A. 1981 Variational principle for two-dimensional incompressible hydrodynamics and quasigeostrophic flows. *Phys. Rev. Lett.* **47**, 1181–1183.

Weissman, M. A. 1978 Nonlinear wave packets in the Kelvin-Helmholtz instability. *Phil. Trans. R. Soc. Lond.* A **290**, 639–685.

Whitham, G. B. 1974 *Linear and Nonlinear Waves*. Wiley.

Yakubovich, E. I. 1974 On the non-degenerated interaction of parametrically linked waves. *Izv. VUZov, Radiofizika*, **17**, 627–629 (in Russian).

Zakharov, V. E. 1974 Hamiltonian formalism for waves in nonlinear dispersive media. *Izv. VUZov, Radiofizika*, **17**, 431–453 (in Russian).

Zakharov, V. E. 1976 Exact solutions to the problem of the parametric interaction of three-dimensional wave packets. *Sov. Phys. Dokl.* **21**, 322–323.

Zakharov, V. E., L'vov, V. S. & Falkovich, G. 1992 *Kolmogorov Spectra of Turbulence I. Wave Turbulence*. Springer.